Particle Physics: Special Relativity

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Week 3: February 10, 2017
Columbia University Science Honors Program

COLUMBIA UNIVERSITY
IN THE CITY OF NEW YORK

NEVIS LABORATORIES

Course policies

- Classes from 10:00 AM to 12:30 PM (10 min break at ~ 11:10 AM).
- **Attendance record counts.**
  - Up to four absences
  - Lateness or leaving early counts as half-absence
  - Send email notifications of all absences to shpattendance@columbia.edu.
- Please, no cell phones during class
- **Please, ask questions!**
- Lecture materials + Research Opportunities + Resources to become a particle physicist

  https://twiki.nevis.columbia.edu/twiki/bin/view/Main/ScienceHonorsProgram

Schedule

Month	Day	Lecture	Teacher
~~January~~	~~27~~	~~Introduction~~	~~Jose~~
February	~~3~~	~~History of Particle Physics~~	~~Jose~~
	10	Special Relativity	Jose
	17	Quantum Mechanics	Jose
	24	Experimental Methods	Cris
March	3	The Standard Model - Overview	Cris
	10	The Standard Model - Limitations	Cris
	17	No classes, Columbia University spring break	
	24	Neutrino Theory	Cris
	31	No classes, Easter and Passover weekend	
April	7	Neutrino Experiment	Jose
	14	LHC and Experiments	Ines
	21	No classes, SHP break	
	28	The Higgs Boson and Beyond	Ines
May	5	Particle Cosmology	Cris

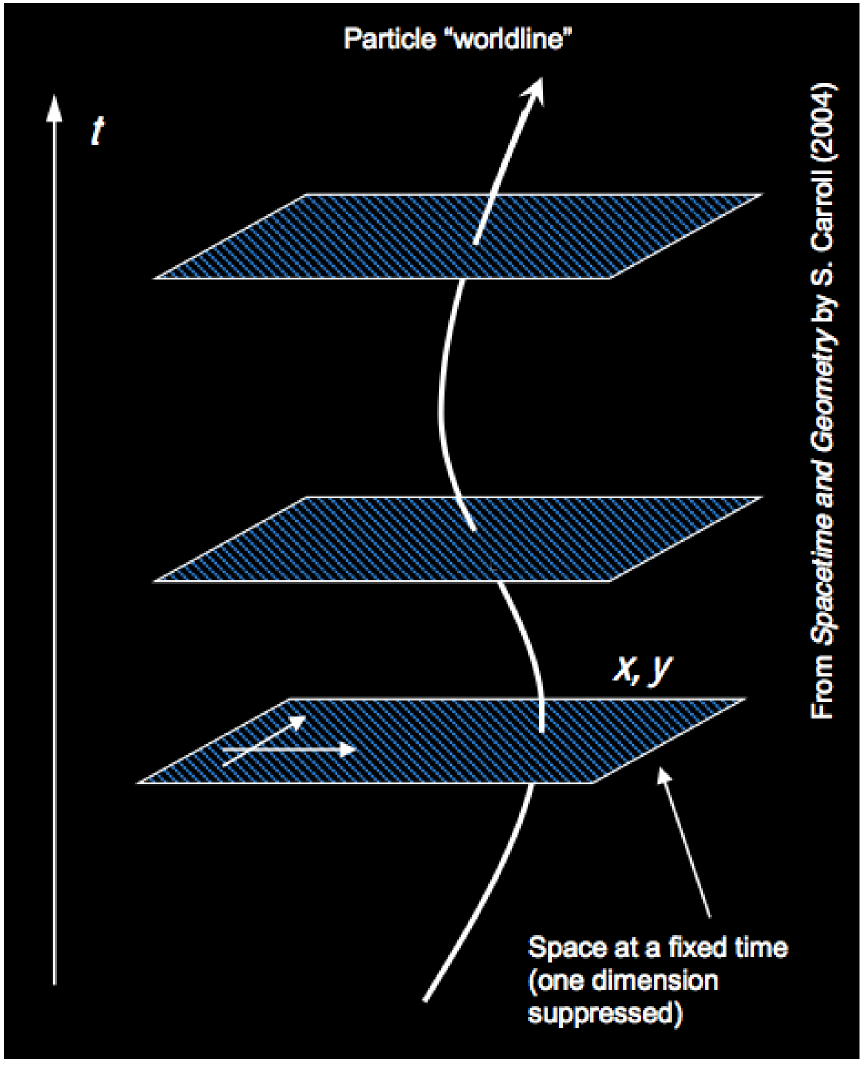
Special Relativity

Relativistic mechanics

- What's wrong with classical mechanics?
  - We will see that classical mechanics is only valid in the limiting case where $\boldsymbol{v} \ll \boldsymbol{c}$.
    This is generally the case for everyday observables.
  - However, this is not the case for Particle Physics since particles are traveling close to the speed of light in most of the times.
    In that case, classical mechanics fails to describe their behavior.
    To properly describe particle kinematics, and particle dynamics, we need relativistic mechanics.

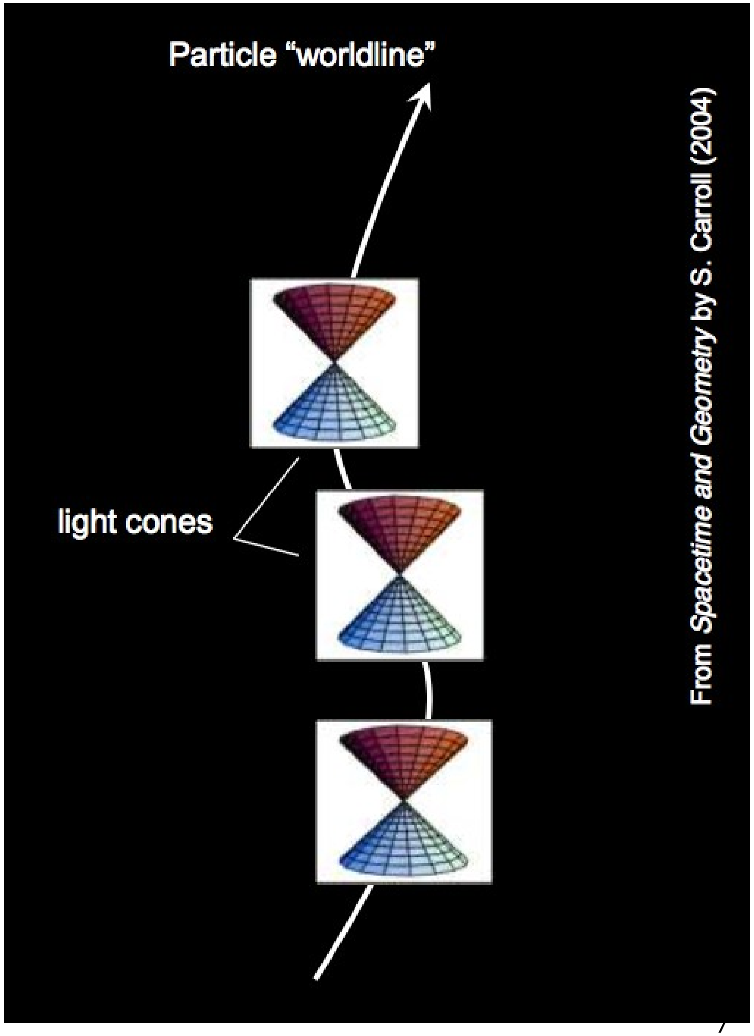
The notion of spacetime

- Spacetime in Newtonian mechanics (“the world, as experienced by us;” v << c):
  - time is universal.
  - space can be cut into distinct “slices” at different moments in time.
  - particles must move forward in time, but can move through space in any direction.
  - all observers agree whether two events at different points in space occur at the same moment of time.



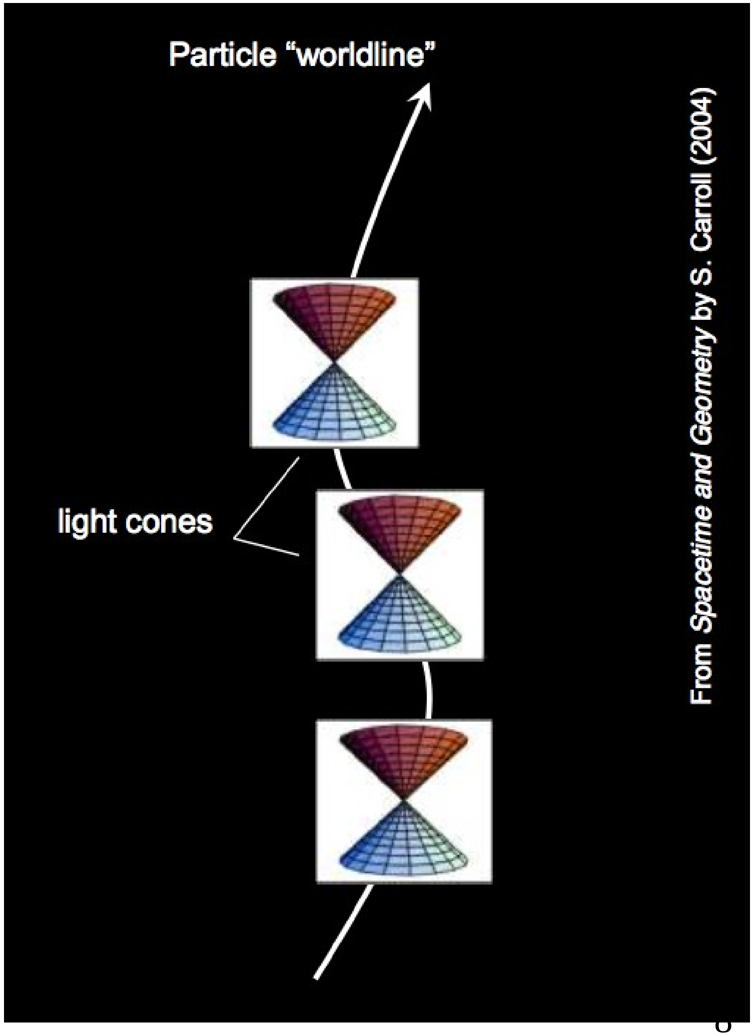
The notion of spacetime

- Spacetime in Special Relativity:
  - time is local.
  - observers may not agree that two events occur at the same time.
  - there is no absolute notion of all space at a moment in time.
  - the speed of light is constant, and cannot be surpassed.
  - every event “exists” within a set of allowed trajectories (light cone).



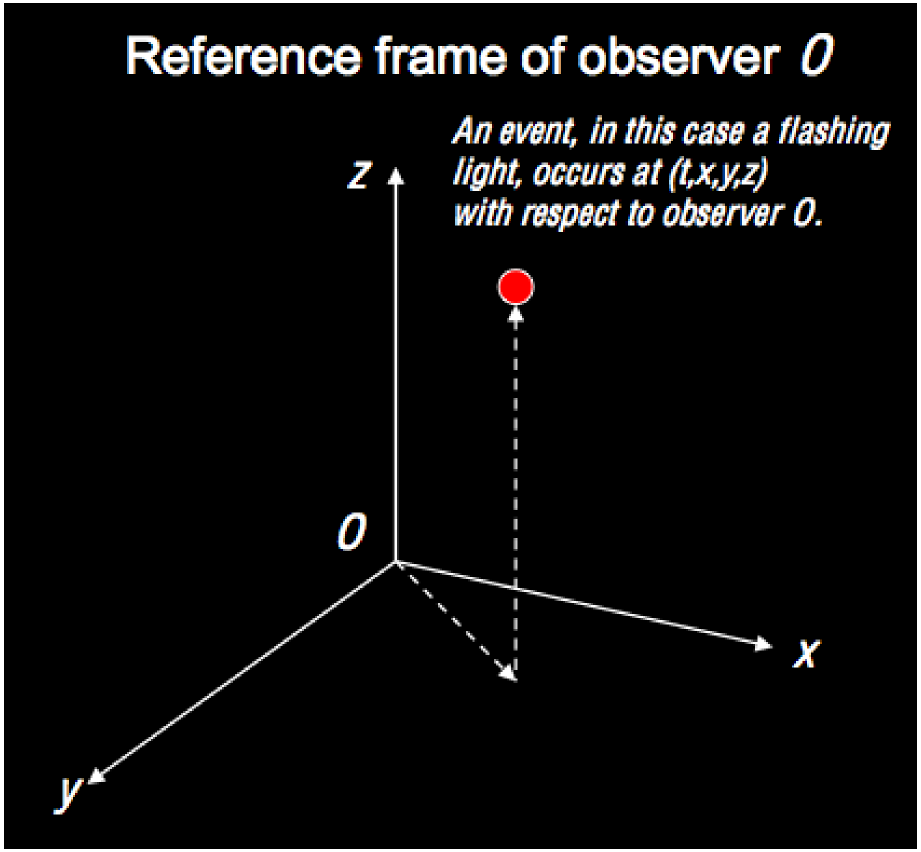
Basic concepts

- **Event:** something that occurs at a specified point in space at a specified time.
- **Observer:** someone who witnesses and can describe events (also known as a “frame of reference”)
  - An observer describes events by using **“standard” clocks and rulers** which are at rest with respect to him/her.



Reference frames

- **What do we mean by "an observer is a frame of reference"?**
- An observer O, in our sense of the word, sets up a Cartesian coordinate system for measuring positions *(x,y,z)*. O then places synchronized clocks at every point in space to measure time.
- Using the spatial coordinate system and clocks, O observes events and assigns each one a time stamp *t* and position *(x,y,z)*.



Inertial observer

- **Inertia: From Newton's first law of motion:** an object not subject to any net external force moves at a constant velocity.
- **Inertial observer: isolated objects that are either at rest or move with constant velocity.** Hence, two inertial observers always move at constant velocity with respect to each other.
- All laws of physics, e.g. Newton's Second Law F=ma are valid for inertial observers.
  - E.g. for an observer moving at constant velocity V with respect to some "fixed" point,

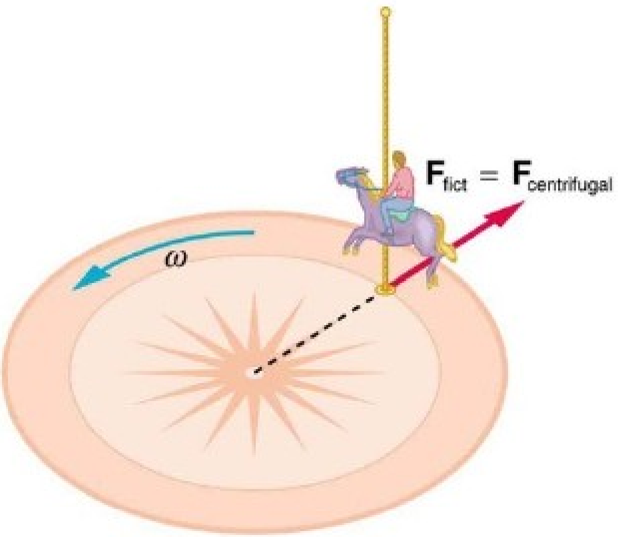
$$v'(t) = v(t) - V$$

$$F = ma' = m\frac{\Delta v'(t)}{\Delta t} = m\frac{\Delta\left(v(t) - V\right)}{\Delta t} = m\frac{\Delta v(t)}{\Delta t} - 0$$

$$= ma$$

Non-Inertial observer

- An observer undergoing acceleration is NOT inertial.
- Accelerating observers feel the influence of "pseudo-forces", resulting in changes to Newton's 2nd Law
- Example: an observer on a merry-go-round spinning at angular velocity ω will perceive that straight-line trajectories bend, and conclude that objects in his/her reference frame are affected by a Coriolis force:



Merry-go-round's rotating frame of reference

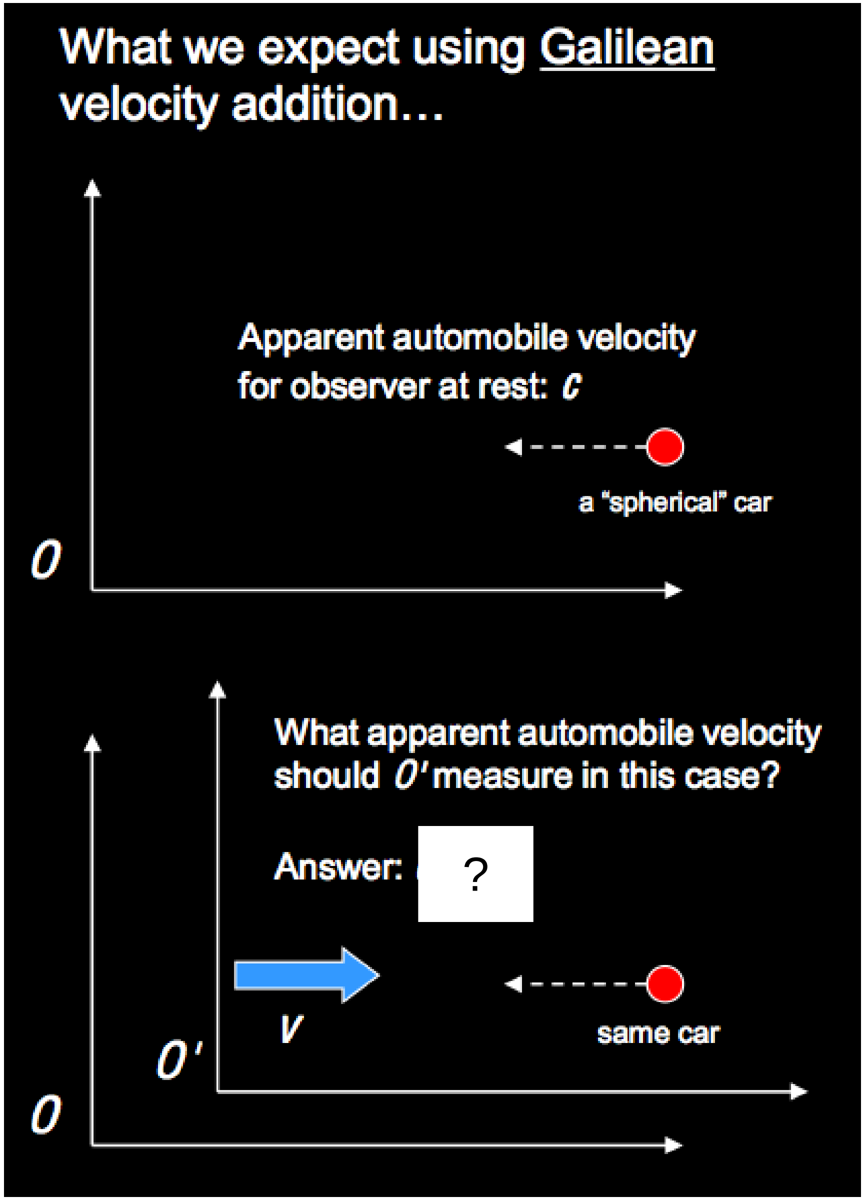
$$\vec{F} = -2m(\vec{\omega} \times \vec{v})$$

Postulates of special relativity

- In 1905, A. Einstein published two papers on special relativity, as well as a paper on the photoelectric effect (Nobel Prize 1921) and Brownian motion (the physics of particles suspended in a fluid).
- All of Einstein's conclusions in special relativity were based on only two simple postulates:
  (1) The laws of physics are the same in all inertial reference frames. (Old idea, dates to Galileo).
  (2) **All inertial observers measure the same speed c for light in a vacuum, independent of the motion of the light source.**

Postulates of special relativity

- The constancy of the speed of light is counter-intuitive, because this is not how "ordinary" objects behave.
- Example: Imagine observing an oncoming car that moves at speed c.
  - We expect a moving observer to measure a different value for c than a stationary one. According to SR, however, for light we always measure the same c, regardless of our motion!



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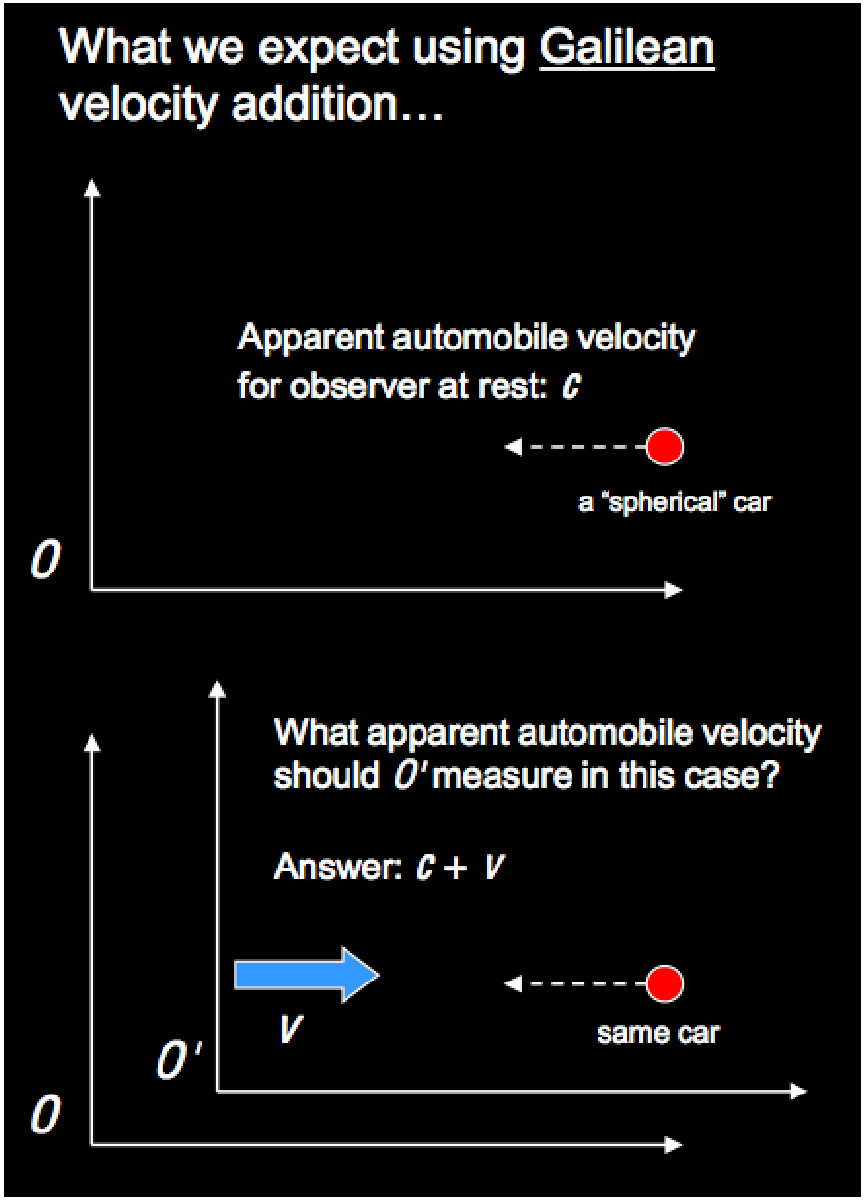
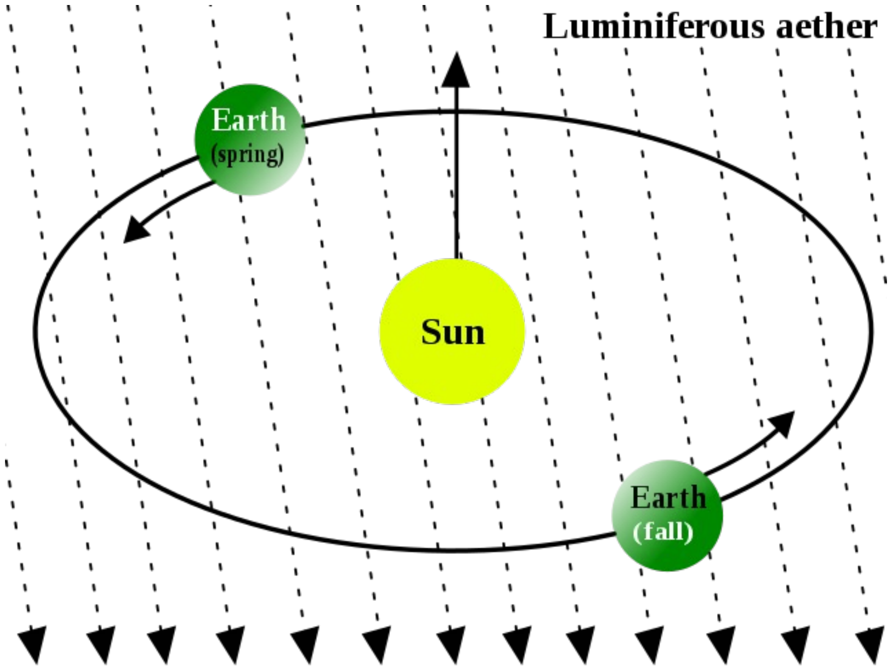


Constancy of speed of light

- The universality of c was first determined experimentally in 1887 by **A. Michelson and E. Morley**.
- At the time, it was believed that light propagated through a medium called the *ether* – physicists didn't think light self-propagated through empty space.
- Using an interferometer, Michelson and Morley expected to see the effect of changes in the speed of light relative to the ether velocity.



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Constancy of speed of light

(Assuming the speed of light is not universal)

Earth's motion through the ether creates an "ether wind" of speed v. Light moving "upwind" should have a speed c-v, and "downwind" c+v. By rotating the interferometer, we should observe a change in the light beams' interference pattern due to the changing beam speed.

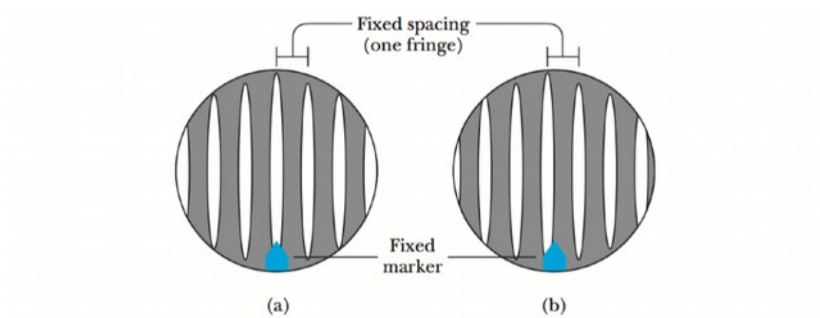
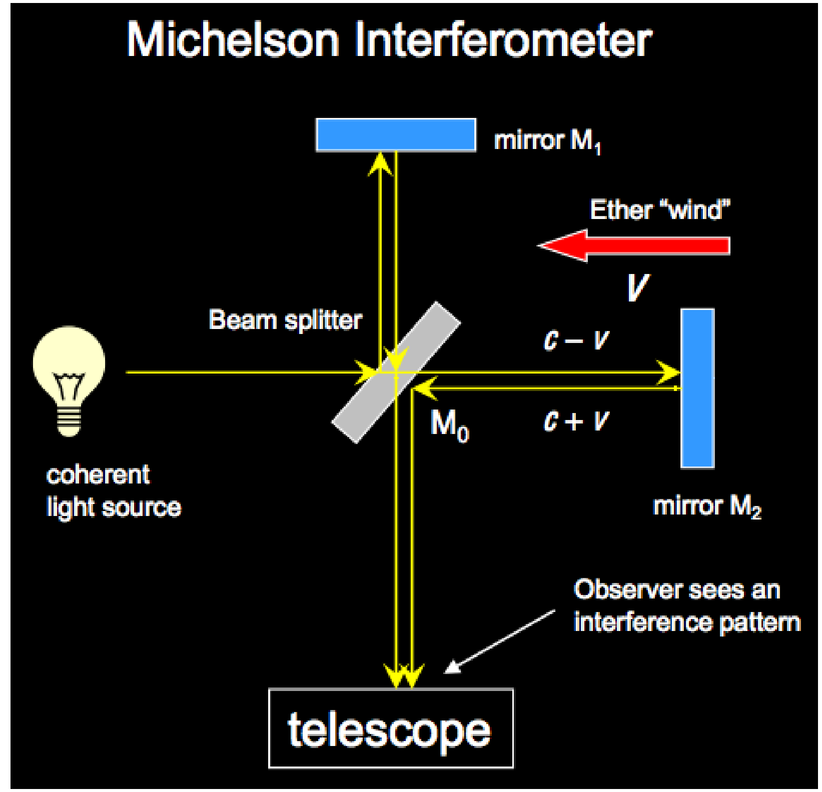
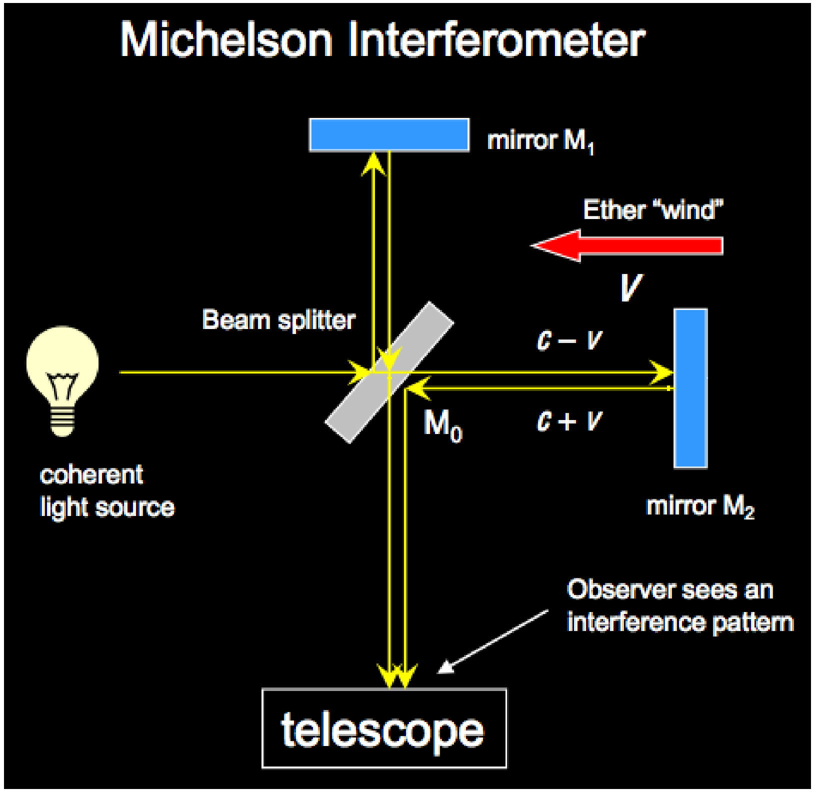


Figure 1.6 Interference fringe schematic showing (a) fringes before rotation and (b) expected fringe shift after a rotation of the interferometer by 90°.



Constancy of speed of light

- In fact, they saw no such effect during repeated trials over several years. The simplest way to explain the result is to assume that there is no ether, and **c is constant in all inertial frames.**



In 1983, the meter was redefined in the International System of Units (SI) as the distance traveled by light in vacuum in 1⁄299,792,458 of a second.

Homework: measure the speed of light!

- https://www.youtube.com/watch?v=kpB1wezpJeE

Become an experimental physicist!

- Repeat the experiment multiple times. How the values are distributed?
- Test multiple setups: what material works better? (chocolate, cheese, marshmallow, licorice...)
- Optimize methodology: what microwave settings work better?

Implications of the postulates

- Einstein developed a series of “thought experiments” that illustrate the interesting consequences of the universality of c. These can be summarized as:
  1) The illusion of simultaneity
  2) Time dilation
  3) Lorentz (length) contraction
  4) Velocity addition

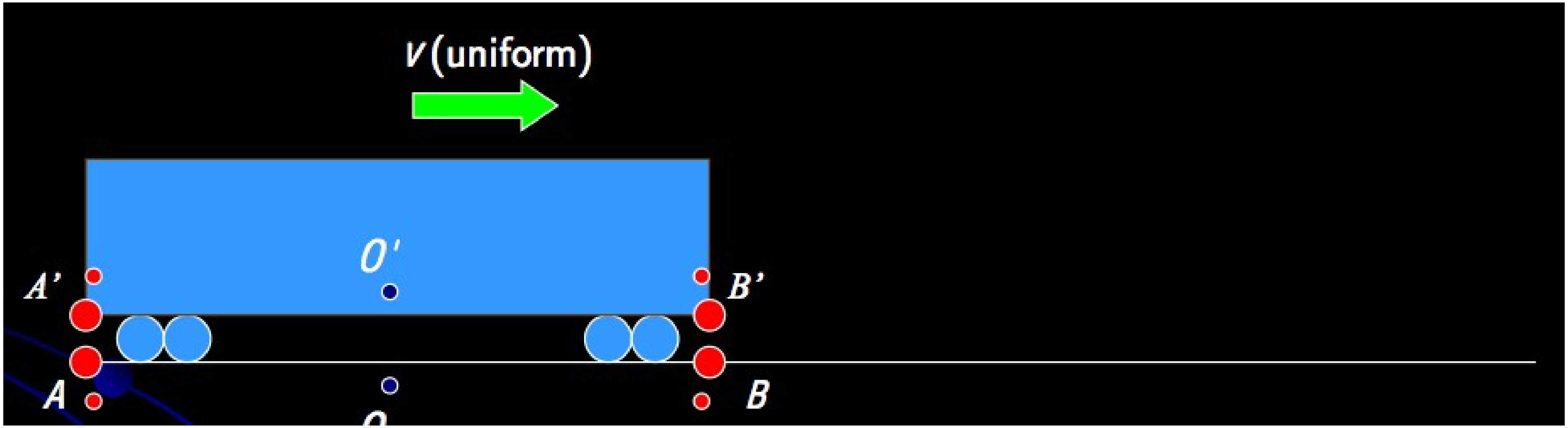
  As we go through Einstein’s examples, keep in mind that these results may seem a little counterintuitive.

The relativity of simultaneity

- An observer O calls two events simultaneous if they occur at the same time in his/her coordinates.
- Interestingly, if the two events do not occur at the same *position* in frame O, then they will not appear simultaneous to a moving observer O'.
- In other words, events that are simultaneous in one inertial system are not necessarily simultaneous in others. Simultaneity is not an absolute concept, but one that depends on the state of motion of the observer.
- Again, this follows from the fact that c is the same in all inertial frames…

The relativity of simultaneity

- A demonstration: Einstein's thought experiment. Flashed light from two ends of a moving boxcar is viewed by two observers. One sits inside the boxcar, in the middle, and the other is stationary (outside, but also in the middle). The lights are set up such that sources A and A' flash at the same time, and B and B' flash at the same time.



- Suppose the flashes from A and B appear simultaneous to O. Do the A' and B' flashes appear simultaneous to O'?

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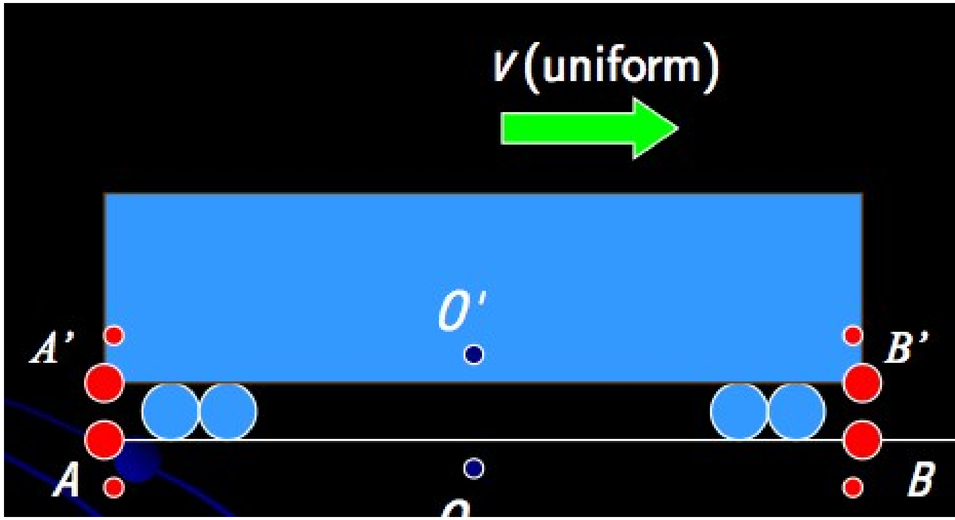


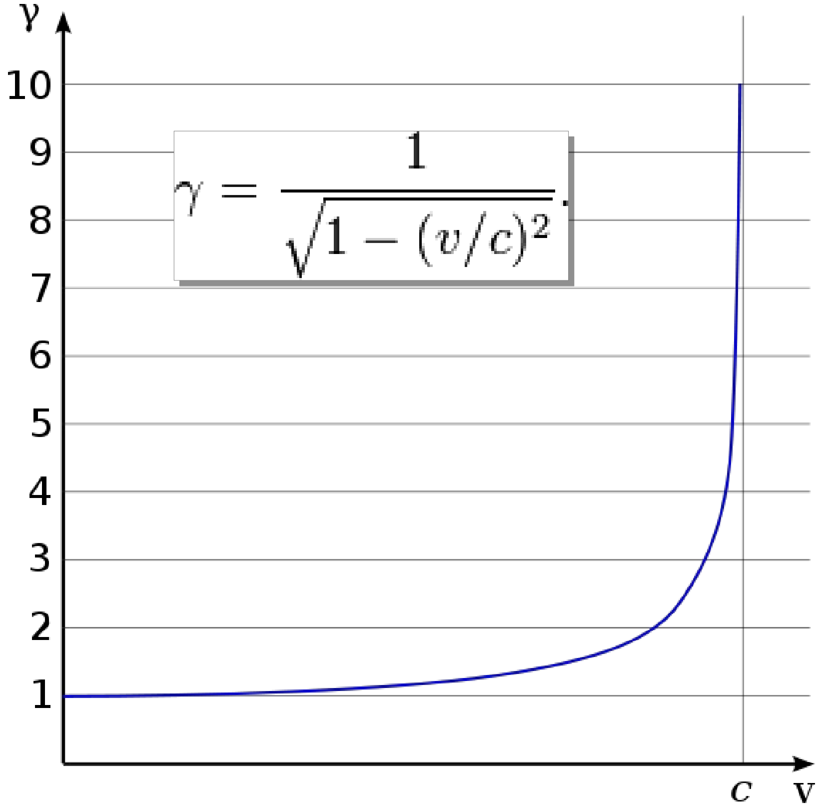
In this case, observer O’ sees the light from B’ first, since the flash emitted by the moving sources A’ and B’ must travel at the same speed as that emitted by A and B.

- This is not what Galilean/Newtonian physics predicts.

Time dilation

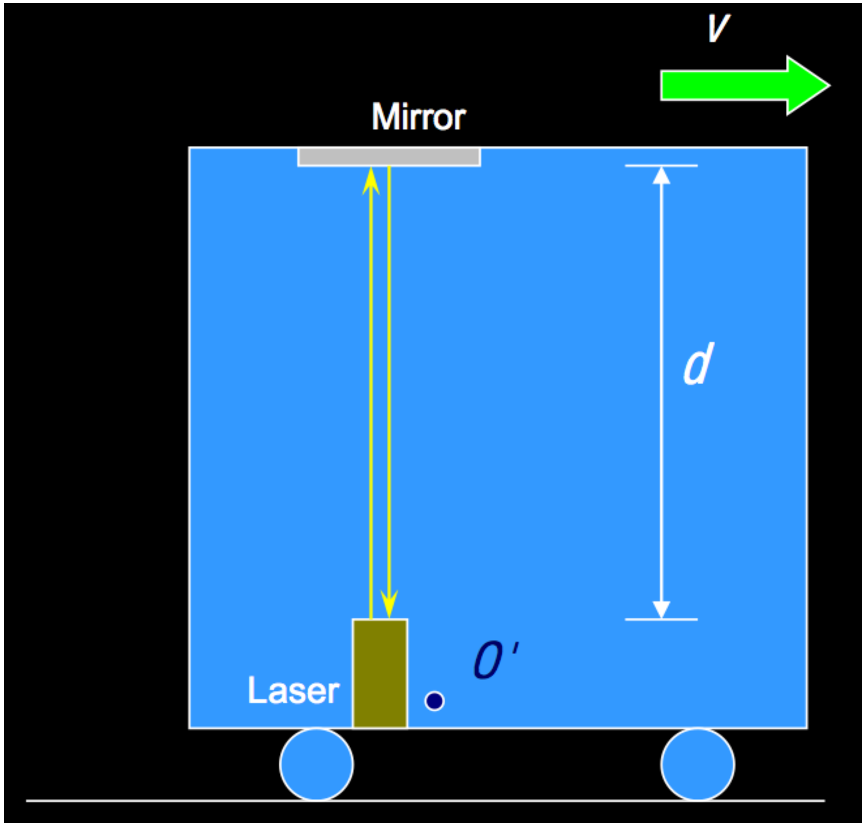
- Time dilation reflects the fact that observers in different inertial frames always measure different time intervals between a pair of events.
- Specifically, an observer O at rest will measure a *longer* elapsed time between a pair of events than an observer O' in motion, i.e. **moving clocks tick more slowly than stationary clocks!**
- The amount by which the observer at rest sees the time interval "dilated" with respect to the measurement by O' is given by the factor called the Lorentz factor γ:

$$\Delta t = \frac{\Delta t'}{\sqrt{1 - v^2/c^2}} = \gamma \Delta t'$$



Time dilation

- Another thought experiment
- Suppose an observer O’ is at rest in a moving vehicle. She has a laser which she aims at a mirror on the ceiling.
- According to O’, how long does it take the laser light to reach the ceiling of the car and bounce back to the ground?



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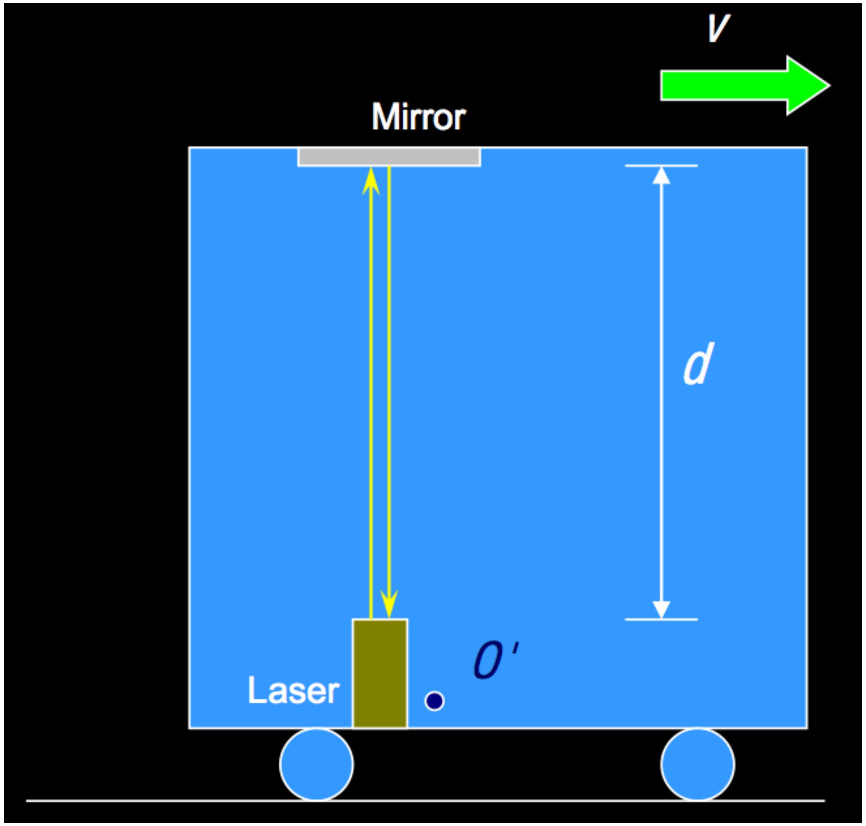
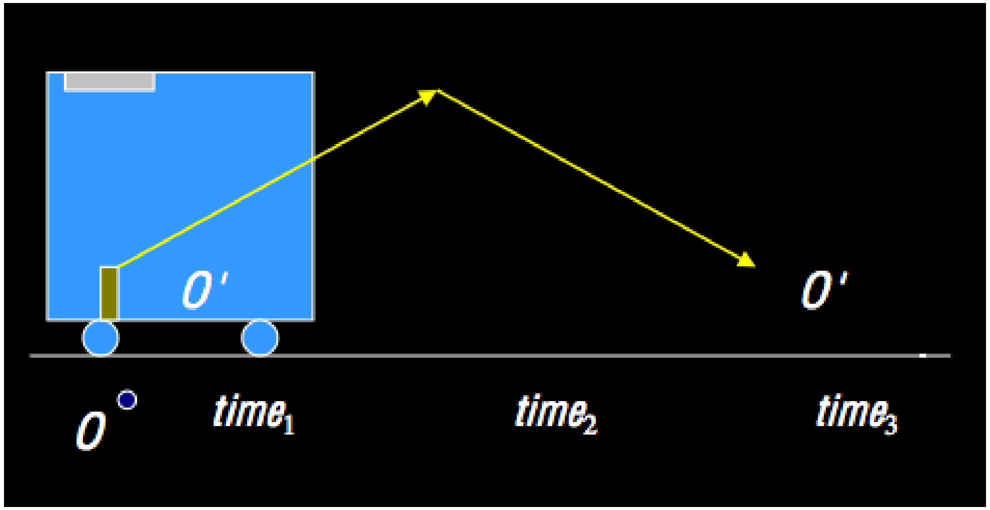


$$\Delta t' = \frac{\text{distance traveled}}{\text{speed}} = \frac{2d}{c}$$

Time dilation

- An observer O *outside* the car sees that it takes time Δt for the laser light to hit the mirror and come back. In that time, the car will have moved a distance $v\Delta t$ according to O.
- In other words, due to the motion of the vehicle, O sees that the laser light must leave the laser at an angle if it is to hit the mirror.
- Show that $\Delta t = \gamma \Delta t'$



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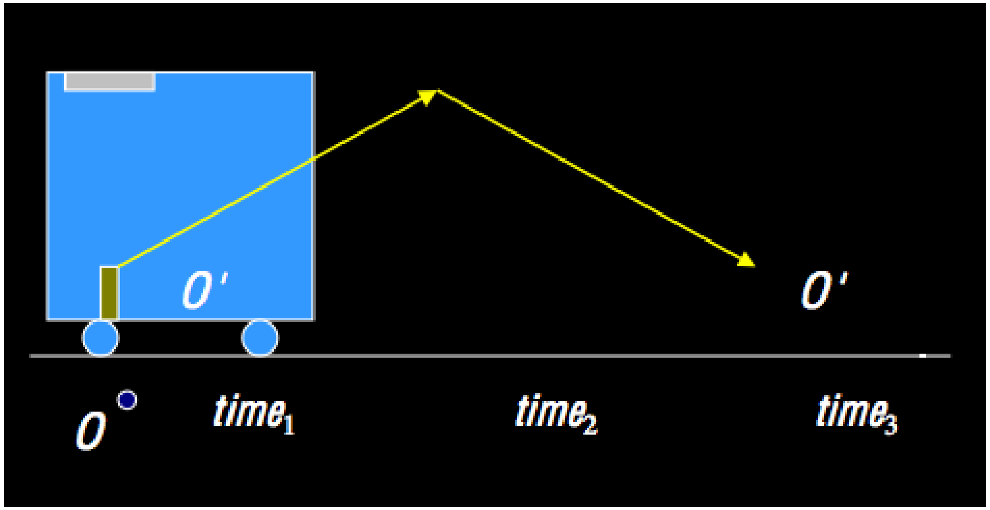
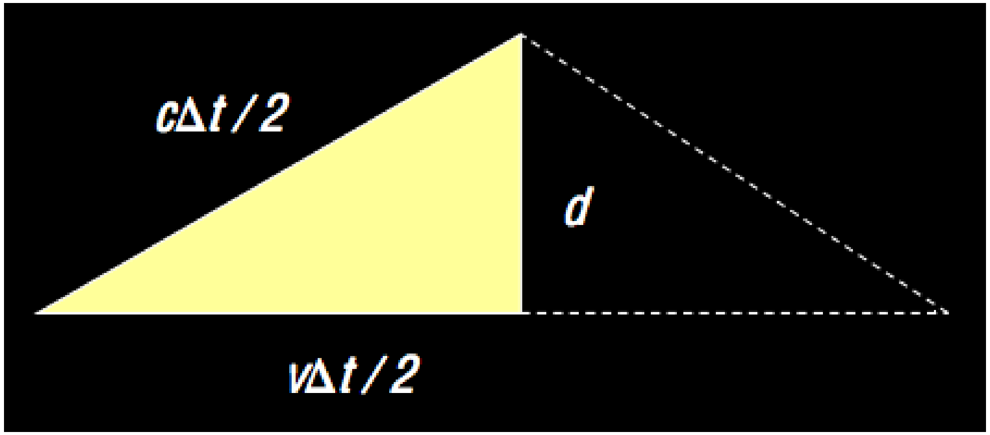
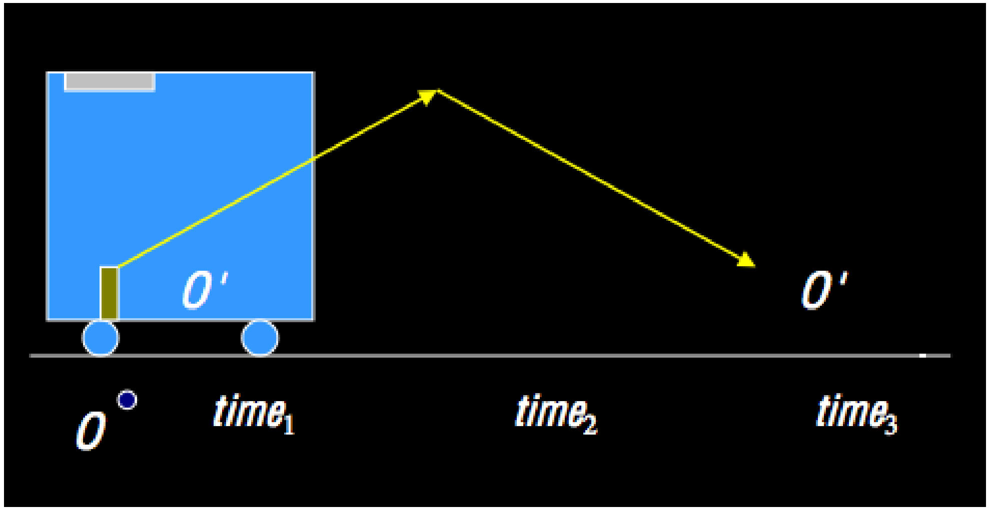




Hint: Use right triangle

Time dilation

- An observer O *outside* the car sees that it takes time Δt for the laser light to hit the mirror and come back. In that time, the car will have moved a distance vΔt according to O.
- In other words, due to the motion of the vehicle, O sees that the laser light must leave the laser at an angle if it is to hit the mirror.
- Show that Δt=γΔt’



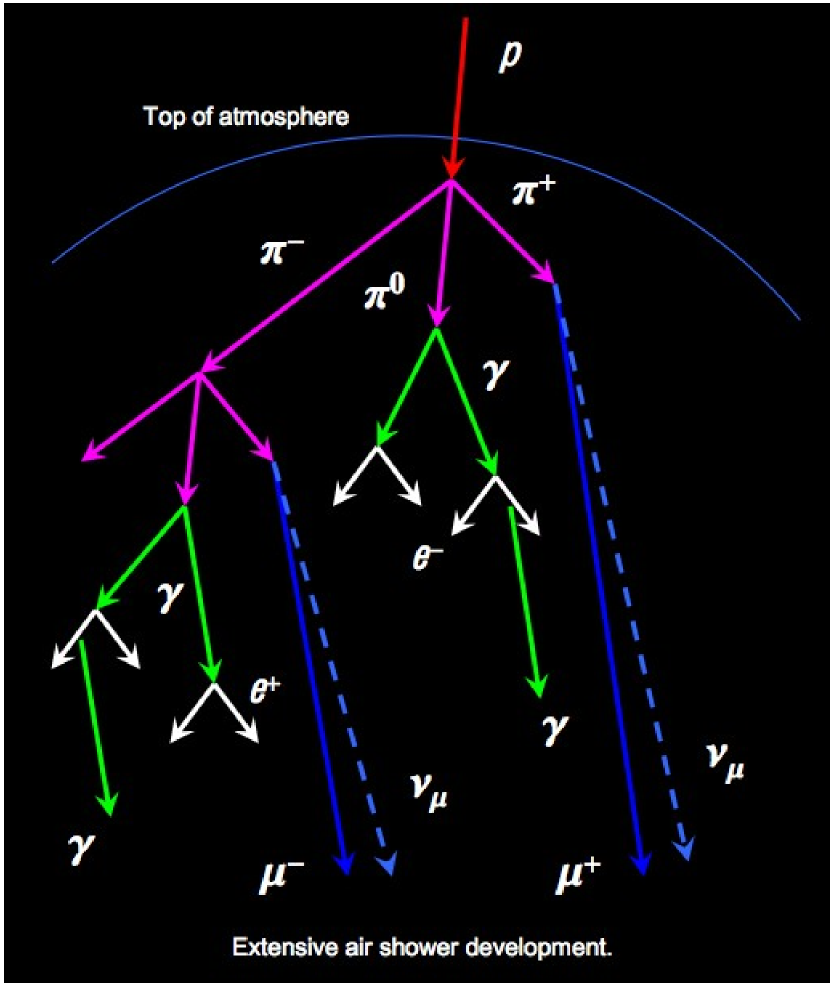
Solution $$\left(\frac{c\Delta t}{2}\right)^2 = \left(\frac{v\Delta t}{2}\right)^2 + d^2$$

$$\Delta t = \frac{2d}{\sqrt{c^2 - v^2}} = \frac{2d/c}{\sqrt{1 - v^2/c^2}}$$

$$\Delta t = \gamma \Delta t'$$

Time dilation in practice

- Recall our mention of cosmic ray air showers…
- Relativistic nuclei strike the atmosphere, causing a huge cascade of high energy decay products. Many of these products are detected at Earth's surface. However, most of them (like π's and μ's) are very unstable and short-lived.
- How do they make it to Earth's surface?



Time dilation in practice

- Naively:
- The mean lifetime of the muon (in its rest frame) is 2.2 microseconds.
- Most air shower muons are generated high in the atmosphere (~8 km altitude). If they travel at 99.9% of the speed of light c, should they make it to Earth from that altitude?

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Time dilation in practice

- Naively:
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- Most air shower muons are generated high in the atmosphere (~8 km altitude). If they travel at 99.9% of the speed of light c, should they make it to Earth from that altitude?

$$\begin{aligned} \text{Muon range} &= (\text{lifetime}) \times (\text{speed}) \\ &= (2.2 \times 10^{-6}\ \text{s}) \times (0.999c) \\ &\approx 660\ \text{m} \end{aligned}$$

- This suggests that muons should not be able to make it to Earth's surface. But we detect them. Where did the calculation go wrong?

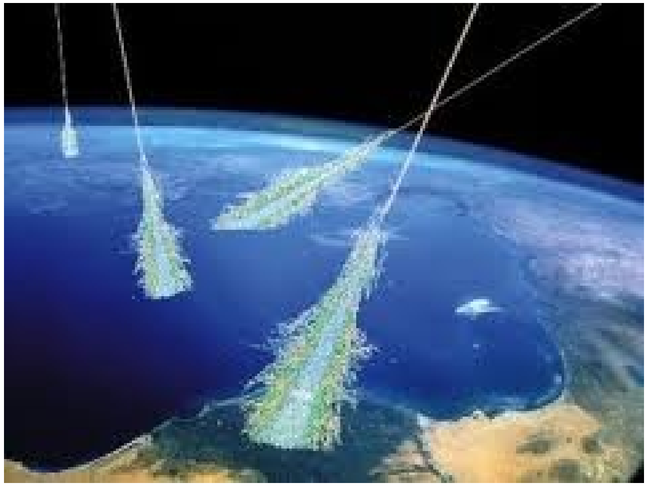
Time dilation in practice

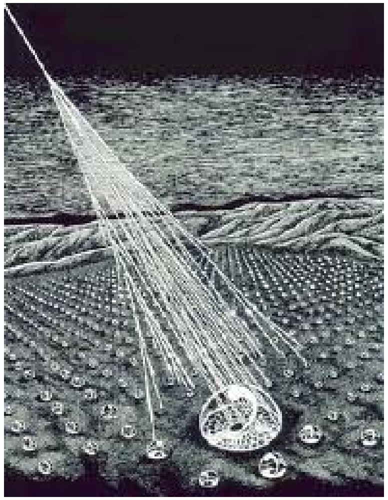
- Accounting for relativity:
- In the lab (the stationary frame), the muon's lifetime undergoes time dilation (a muon's clock ticks slower…).
  Therefore, we have an **effective** lifetime to deal with:

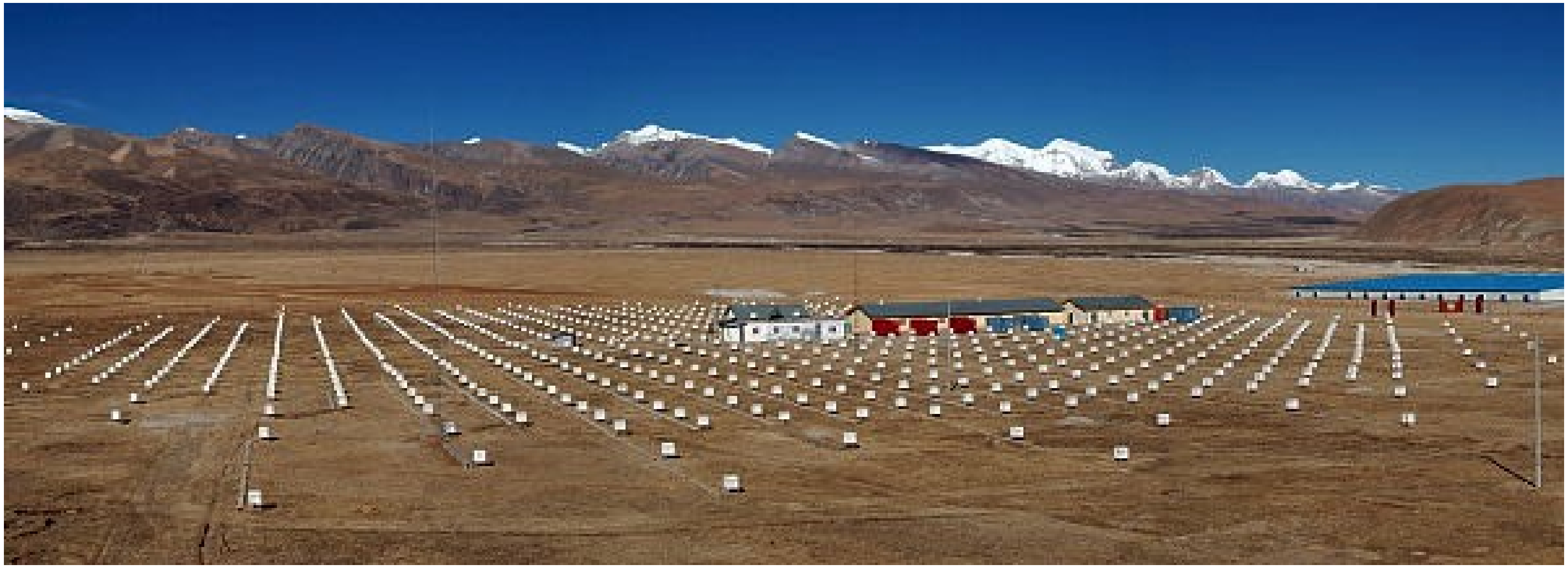
$$\begin{aligned}\text{Muon range} &= \gamma \times (\text{lifetime}) \times (\text{speed}) \\ &= \left(\frac{1}{\sqrt{1-(0.999c/c)^2}}\right) \times (2.2 \times 10^{-6}\ \text{s}) \times (0.999c) \\ &\approx 14.7\ \text{km}\end{aligned}$$

- So the muon can certainly make it to the ground, on average, when we account for relativistic effects.

Cosmic ray experiments





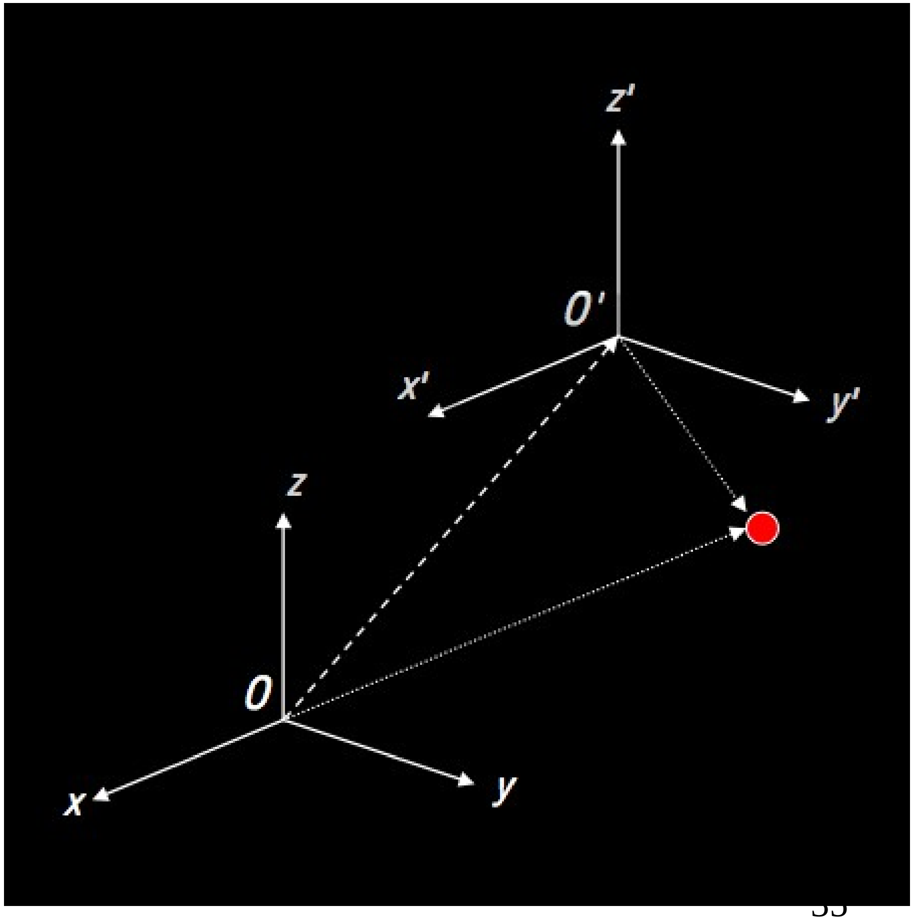


Transformations between reference frames

- Using the postulates of Special Relativity, we can start to work out how to *transform coordinates* between different inertial observers.
  - What is a transformation? It's a mathematical operation that takes us from one inertial observer's coordinate system into another's.
- The set of possible transformations between inertial reference frames are called the **Lorentz Transformations**. They form a group (in the mathematical sense of "group theory").
- The possible Lorentz Transformations:
  - Translations
  - Rotations
  - Boosts.

Translations (fixed displacements)

- In fixed translations, the two observers have different origins, but don't move with respect to each other.
- In this case, the observers' clocks differ by a constant b_0 and their positions differ by a constant vector *b:*



$$\vec{x}' = \vec{x} - \vec{b}$$

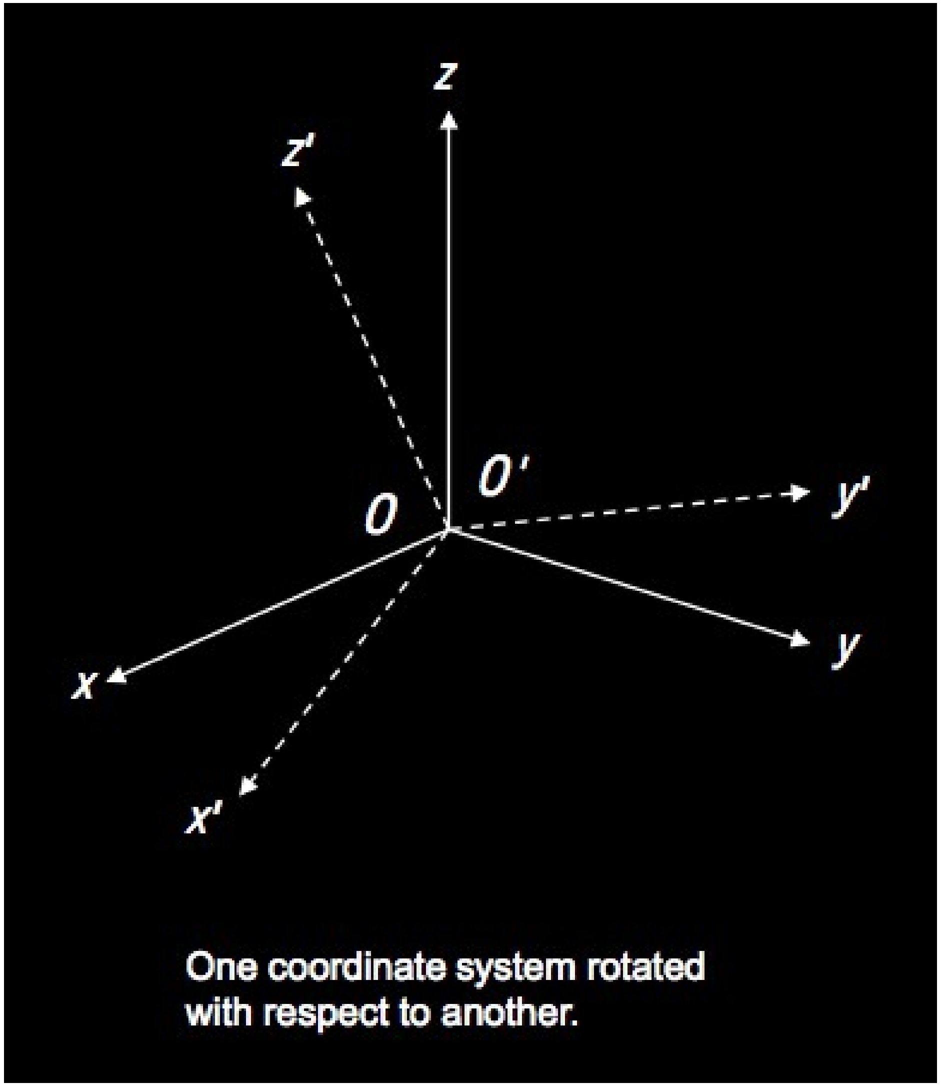
$$t' = t - b_0$$

Rotations (fixed)

- In fixed rotations, the two observers have a common origin and don't move with respect to each other.
- In this case, the observers' coordinates are rotated with respect to each other.
- The spatial transformation can be accomplished with a rotation matrix; measured times are the same:

$$\vec{x}' = \overleftrightarrow{R} \cdot \vec{x}$$

$$t' = t$$



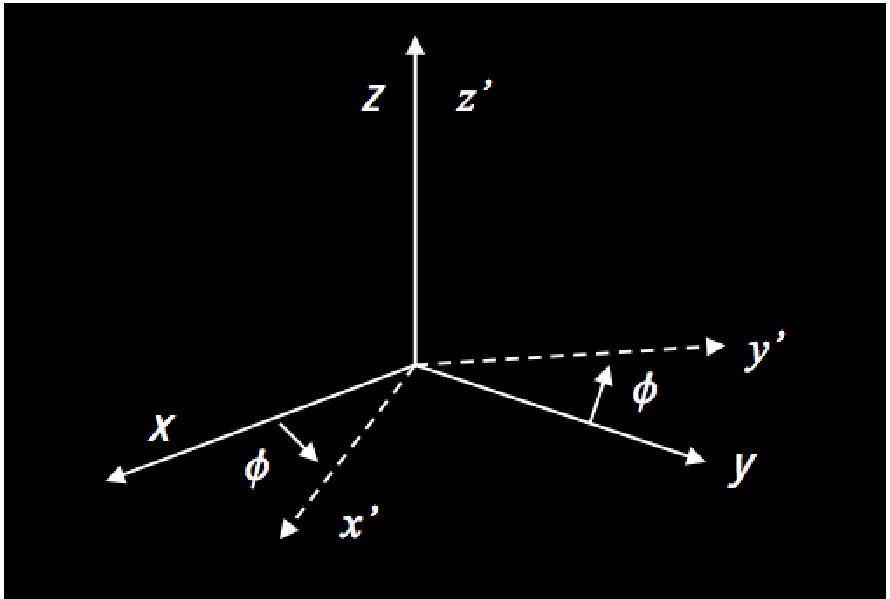
Fixed rotation example

- Consider two observers; they share a common origin and z-axis, but the x-y plane of O' is rotated counterclockwise by an angle of ϕ relative to O.
- Their unit vectors are related by:

$$\begin{pmatrix} \hat{x}' \\ \hat{y}' \\ \hat{z}' \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix},$$

or

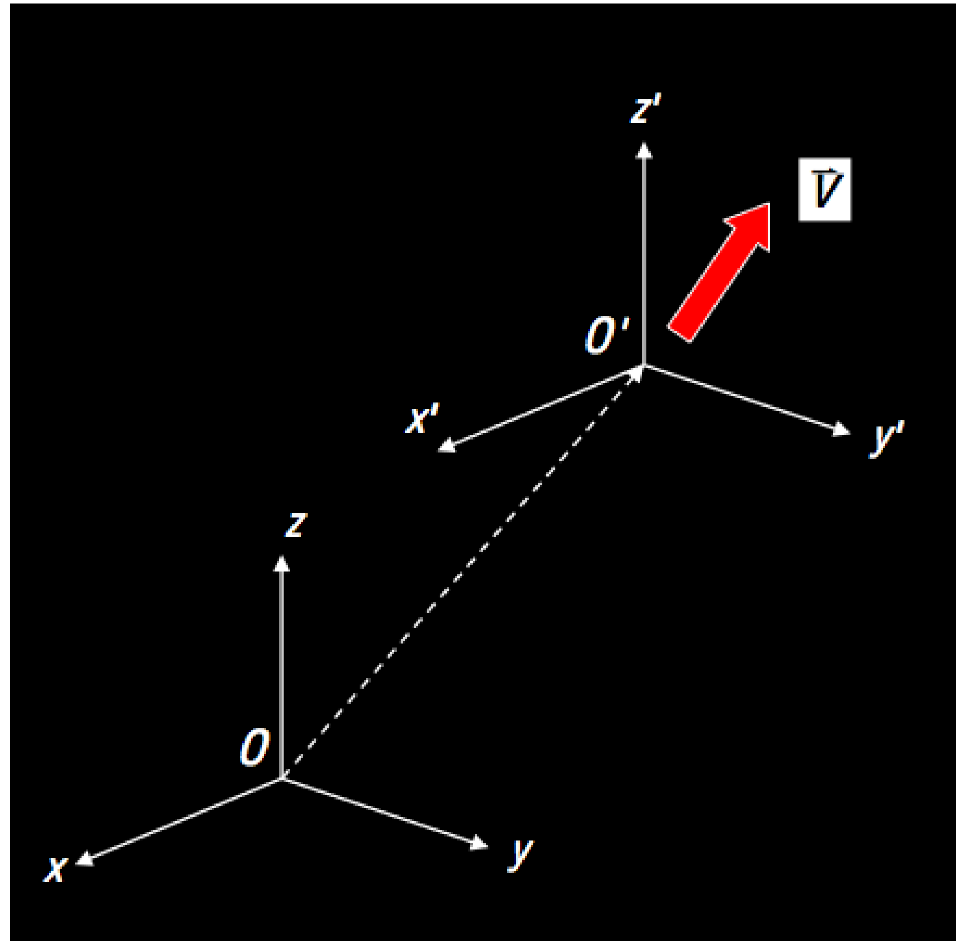
$$\hat{x}' = \hat{x}\cos\phi + \hat{y}\sin\phi$$
$$\hat{y}' = -\hat{x}\sin\phi + \hat{y}\cos\phi$$
$$\hat{z}' = \hat{z}$$



A rotation about the z axis. The unit vectors of the rotated coordinates are related to the original coordinates by a rotation matrix.

Boosts

- In boosts, the two frame axes are aligned, but the **frames move at <u>constant</u> velocity with respect to each other.**
- The origins are chosen here to coincide at time t=0 in both frames.
- The fact that the observers' coordinates are not fixed relative to each other makes boosts more complex than translations and rotations.
- It is in boosts that the constancy of the speed of light plays a big role.



Observers' axes are aligned, but one moves at constant velocity *v* with respect to the other.

Boosts: Galileo vs Lorentz

- Suppose we have two observers O and O'. O is at rest, and O' moves along the x direction with constant velocity v.
- According to Galileo, the transformation between the coordinates of O and O' is pretty simple; but according to Lorentz and Einstein, we get complicated expressions with many factors of c involved: the so-called Lorentz transformations.
- If an event occurs at position (x,y,z) and time t for observer O, what are the spacetime coordinates (x',y',z') and t' measured by O'?
  - Galileo and Lorentz say the following:

Galileo

$$x' = x - vt$$
$$y' = y$$
$$z' = z$$
$$t' = t$$

Lorentz

$$x' = \gamma(x - vt)$$
$$y' = y$$
$$z' = z$$
$$t' = \gamma(t - vx/c^2)$$

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \geq 1$$

Note the Lorentz factor γ in the Lorentz boosts.

Lorentz (length) contraction

- Suppose a moving observer O' puts a rigid "meter" stick along the x' axis: one end is at x'=0, and the other at x'=L'.
- Now an observer O at rest measures the length of the stick at time t=0, when the origins of O and O' are aligned. What will O measure for x'?
- Using the first boost equation $x'=\gamma(x-vt)$ at time t=0, it looks like the lengths are related by:

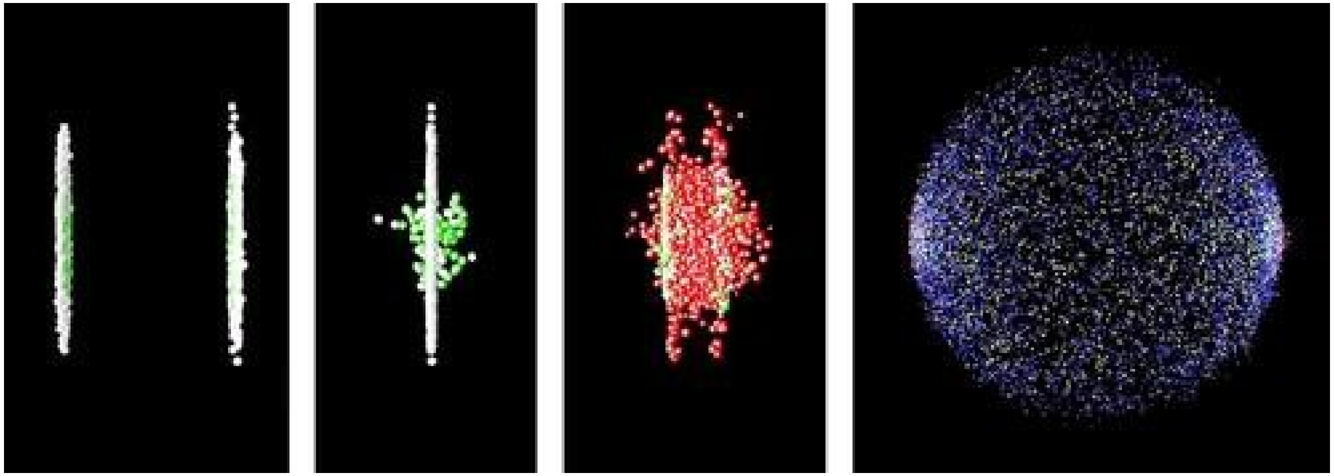
$$\text{(moving) } L' = \gamma \, L \text{ (at rest)}$$

$$L = L' / \gamma$$

- This is the **Lorentz contraction**: if an object has length l when it is at rest, then when it moves with speed v in a direction parallel to its length, an observer at rest will measure its length as the *shorter value* L/γ.

Lorentz contraction

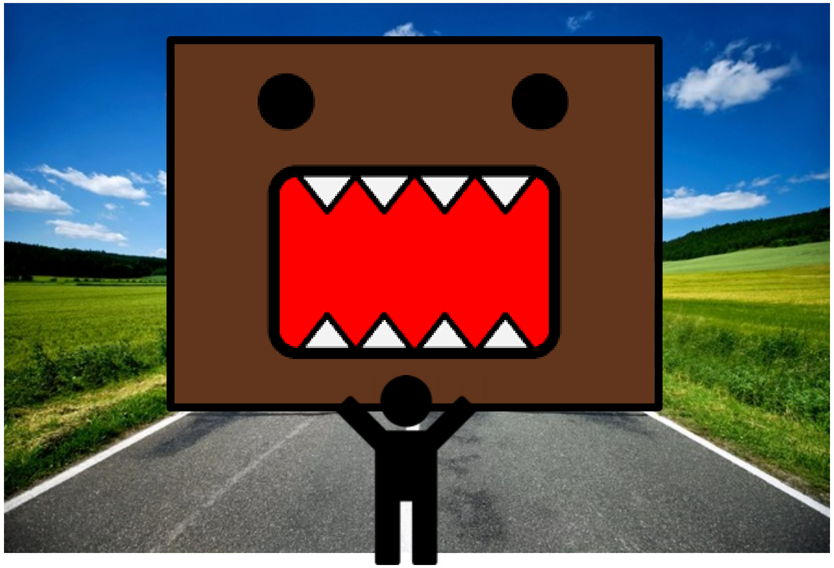
- An example of Lorentz contraction in the case of collisions of two gold nuclei at the RHIC collider at Brookhaven Lab on Long Island:



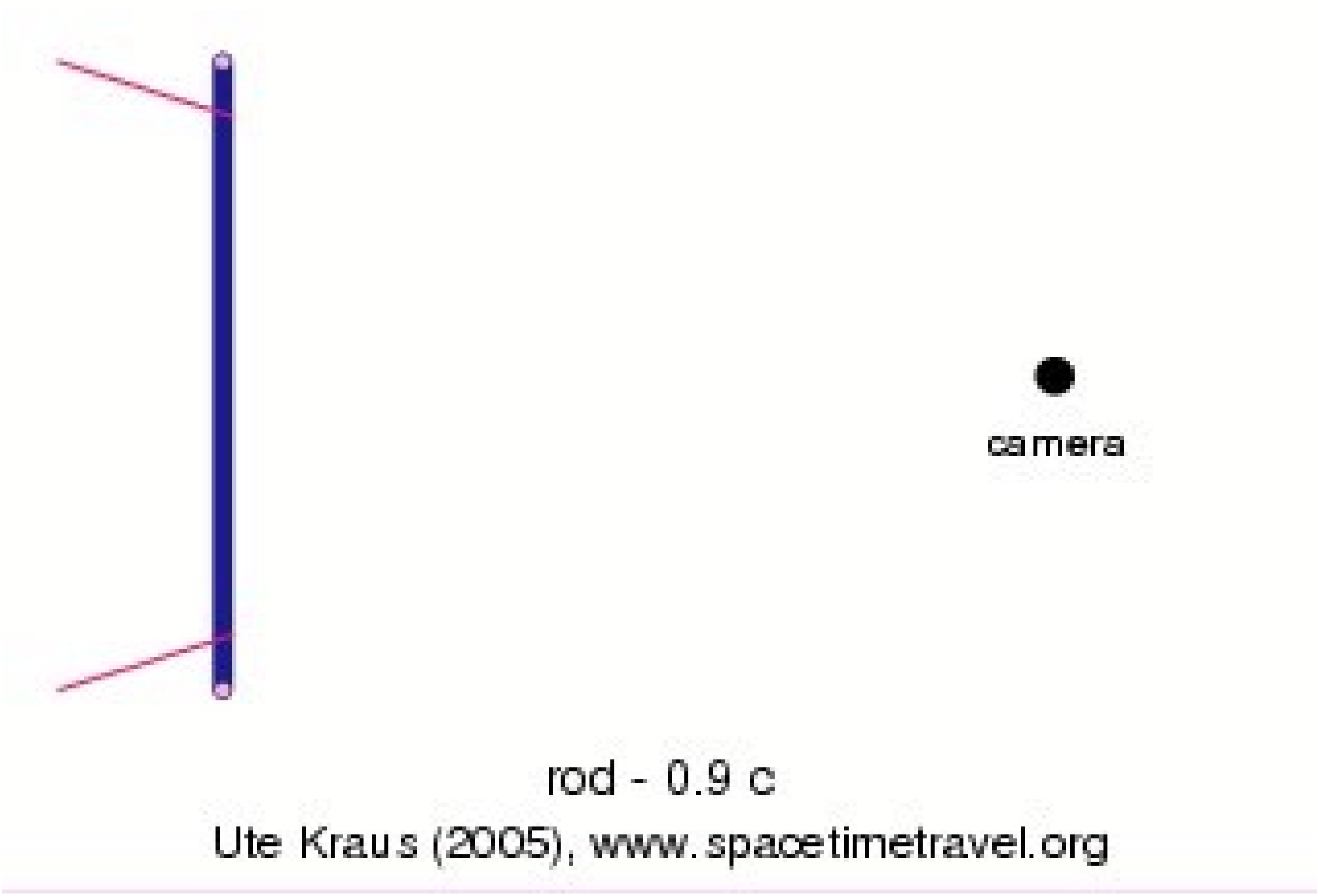
- In typical collisions (200 GeV) nuclei have a Lorentz factor of O(200).

More fun: adding retardation

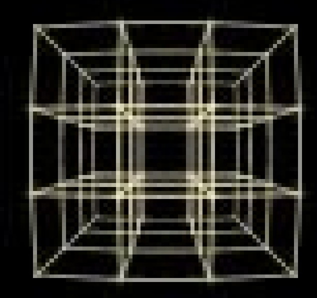
- The stick example to illustrate the Lorentz contraction and the illustration of the heavy ions colliding ignore the finite time the light needs to propagate from the objects to the eyes of the observer.
- You will never see anything as it is but as it was!
  - Your own hand: a few nanoseconds ago.
  - The Sun: 8 min ago.
  - Proxima Centauri (the closest star to the Sun): 4.2 years ago.
- **Exercise**: draw how a relativistic square (v ~ c) looks coming at you!



1-D case: relativistic rod coming at you



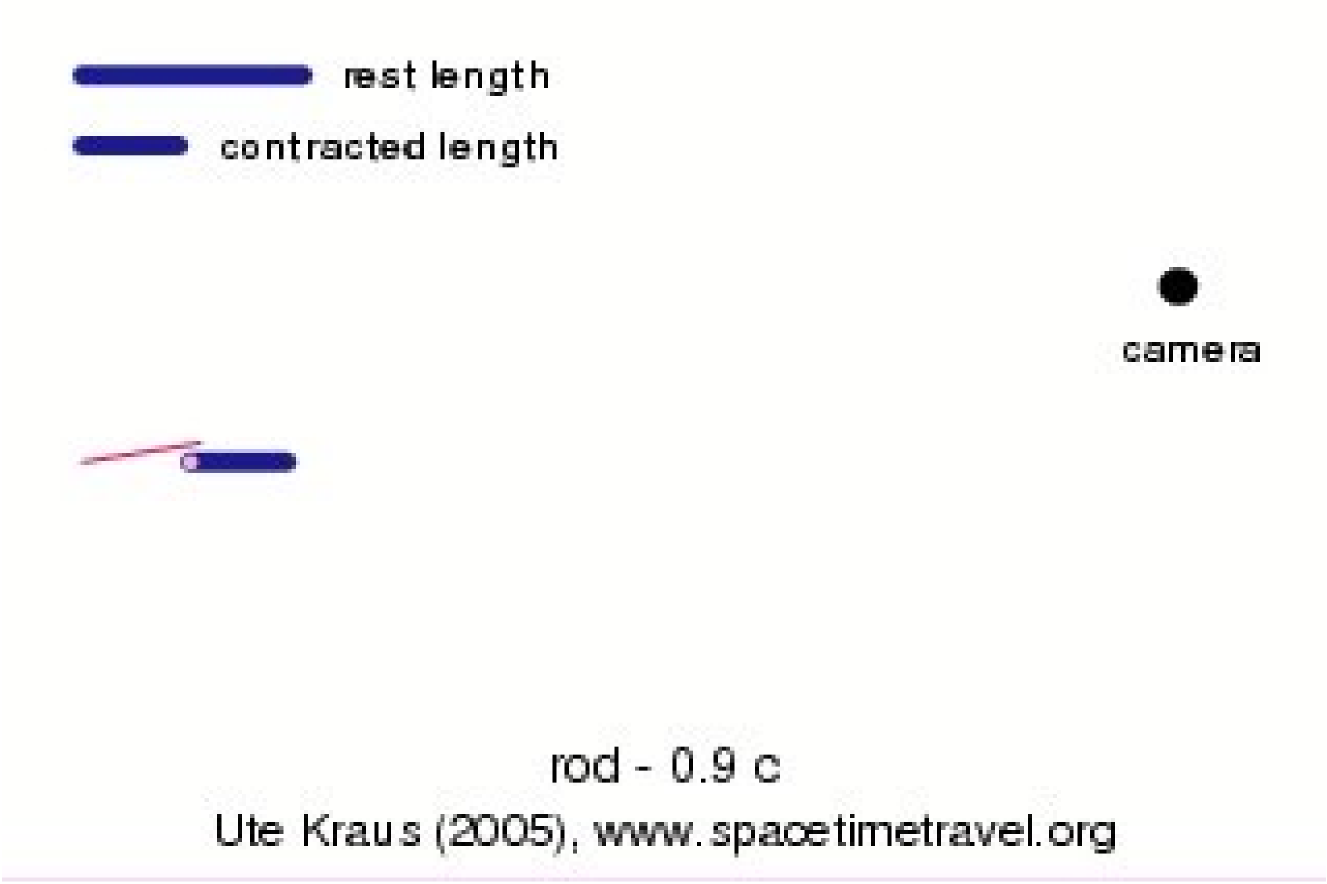
3-D case: relativistic lattice coming at you



cubic lattice - 0.9 c

Ute Kraus (2005), www.spacetimetravel.org

Lorentz contraction...
but does the stick actually look shorter?



Apparent rotation

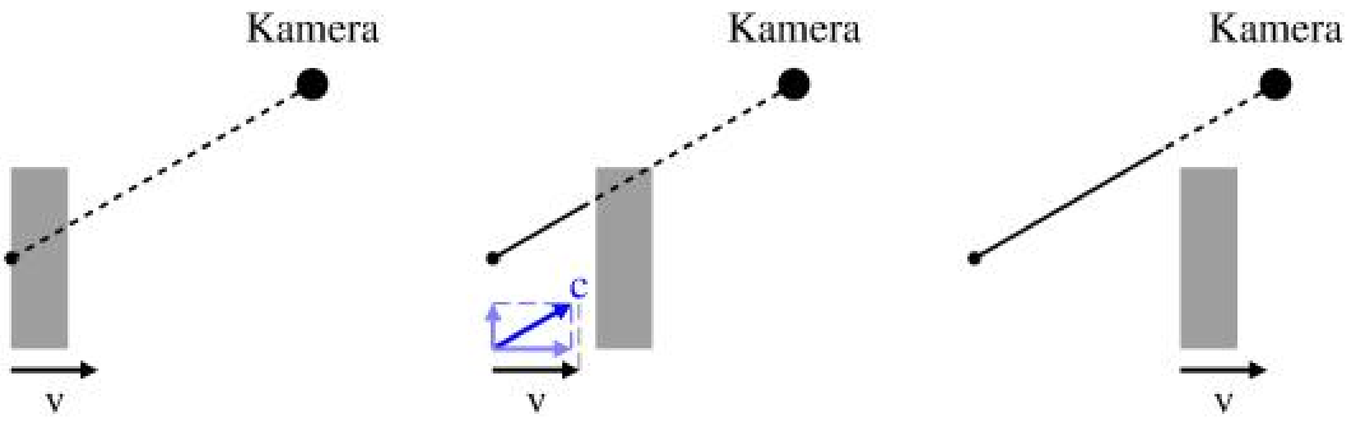
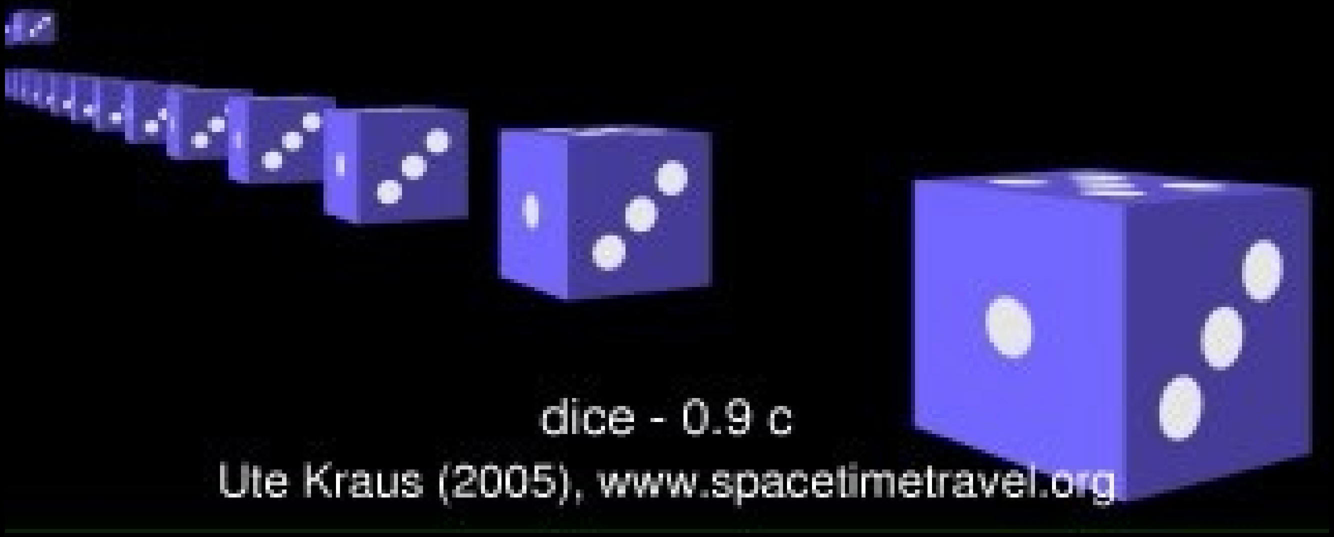


Fig. 6: Why we may see the back side of a cube if the cube is moving fast enough. The photon velocity (blue arrow) has two components (grey), a horizontal one (towards the cube) and a vertical one. If the horizontal component is smaller than the velocity *v* of the cube (black), the photon escapes because the cube outruns the photon and so moves out of the way fast enough. In this figure the cube moves at 95% of the speed of light and, correspondingly, is contracted in the direction of motion to 31% of its rest length. Thin solid lines mark the distance already covered by the light ray, dotted lines indicate the remainder of the light path.

Examples from http://www.spacetimetravel.org/ueberblick/ueberblick1.html
by Ute Kraus and Corvin Zahn

Apparent rotation



Relativistic Tübingen

Velocity addition

- Finally, let's briefly derive the rule for addition of relativistic velocities (we will need to use the boost equations...)
- Suppose a particle is moving in the x direction at speed u' with respect to observer O'. What is its speed u with respect to O?
  - Since the particle travels a distance $\Delta x = \gamma(\Delta x' + v\Delta t')$
  - An "inverse" boost in time $\Delta t = \gamma(\Delta t' + (v/c^2)\Delta x')$
  - The velocity in frame O is:

$$\frac{\Delta x}{\Delta t} = \frac{\Delta x' + v\Delta t'}{\Delta t' + (v/c^2)\Delta x'} = \frac{(\Delta x'/\Delta t') + v}{1 + (v/c^2)(\Delta x'/\Delta t')}$$

  where v is the *relative velocity* of the two inertial frames.

  - Since $u = \Delta x/\Delta t$ and $u' = \Delta x'/\Delta t'$, we get the addition rule:

$$u = \frac{u' + v}{1 + (u' v/c^2)}; \quad \text{compare to} \quad u = u' + v$$

Four-vector notation

- This is a way to simplify notation for all we've talked about so far.
- Soon after Einstein published his papers on Special Relativity, Minkowski noticed that **regarding t and (x,y,z) as simply four coordinates in a 4-D space ("space-time")** really simplified many calculations.
- In this spirit, we can introduce a **position-time *four-vector x***$^{\mu}$, *where* μ*=0,1,2,3, as follows:*

$$x^0 = ct, \quad x^1 = x, \quad x^2 = y, \quad x^3 = z$$

Lorentz boosts in four-vector notation

- In terms of the 4-vector x^μ, a Lorentz boost along the x^1(that is, the x) direction looks like:

$$
\begin{aligned}
x^0{}' &= \gamma(x^0 - \beta x^1) \\
x^1{}' &= \gamma(x^1 - \beta x^0) \\
x^2{}' &= x^2 \\
x^3{}' &= x^3
\end{aligned}
\quad , \quad \text{where } \beta = v/c
$$

- As an exercise, you can show that the above equations recover the Lorentz boosts we discussed earlier.
- FYI, this set of equations also has a very nice and useful matrix form…

Lorentz boosts in matrix form

- Using 4-vectors, we can write the Lorentz boost transformation as a matrix equation

$$\begin{pmatrix} x^{0}{}' \\ x^{1}{}' \\ x^{2}{}' \\ x^{3}{}' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^{0} \\ x^{1} \\ x^{2} \\ x^{3} \end{pmatrix}$$

- Looks very similar to the 3-D rotation! Mathematically, boosts and rotations are actually very close “cousins”. We can understand this connection using the ideas of group theory.

Invariant quantities

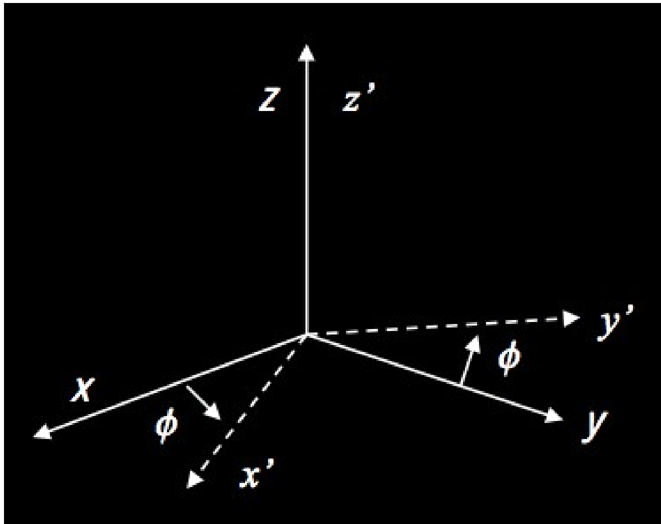
- The utility of 4-vectors comes in when we start to talk about invariant quantities.
- Definition: a quantity is called *invariant* if it has the same value in any inertial system.
- RECALL: the laws of physics are always the same in any inertial coordinate system (this is the definition of an inertial observer). Therefore, these laws are invariants, in a sense.
- The identification of invariants in a system is often the best way to understand its physical behavior.

Example of invariant quantity

- Think of a 3-vector (x,y,z). An example of an invariant is its square magnitude, $r^2=x^2+y^2+z^2$, whose value does not change under coordinate rotations.
- Consider a rotation about the z-axis:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{aligned} r'^2 &= x'^2 + y'^2 + z'^2 \\ &= (x\cos\varphi + y\sin\varphi)^2 + (-x\sin\varphi + y\cos\varphi)^2 + z^2 \\ &= (\cos^2\varphi + \sin^2\varphi)(x^2 + y^2) + z^2 \\ &= r^2 \end{aligned}$$



4-vector scalar product

- The quantity Δs^2, given by:

$$\Delta s^2 = x^0 x^0 - x^1 x^1 - x^2 x^2 - x^3 x^3 = x^0 x^0 - \vec{x} \cdot \vec{x}$$

$$= (ct)^2 - x^2$$

- is called the scalar product of x^μ with itself. It is an invariant, i.e., it **has the same value in any coordinate system** (just like any scalar). This spacetime interval is often called the *proper length.*
- To denote the scalar product of two arbitrary 4-vectors a^μ and b^μ, it is convenient to drop the Greek index and just write:

$$a \cdot b = a^0 b^0 - \vec{a} \cdot \vec{b}$$

  - In this case, the 4-vectors a and b are distinguished from their spatial 3-vector components by the little arrow overbar.

4-vector scalar product

- Terminology: any arbitrary 4-vector a^μ can be classified by the sign of its scalar product a^2:

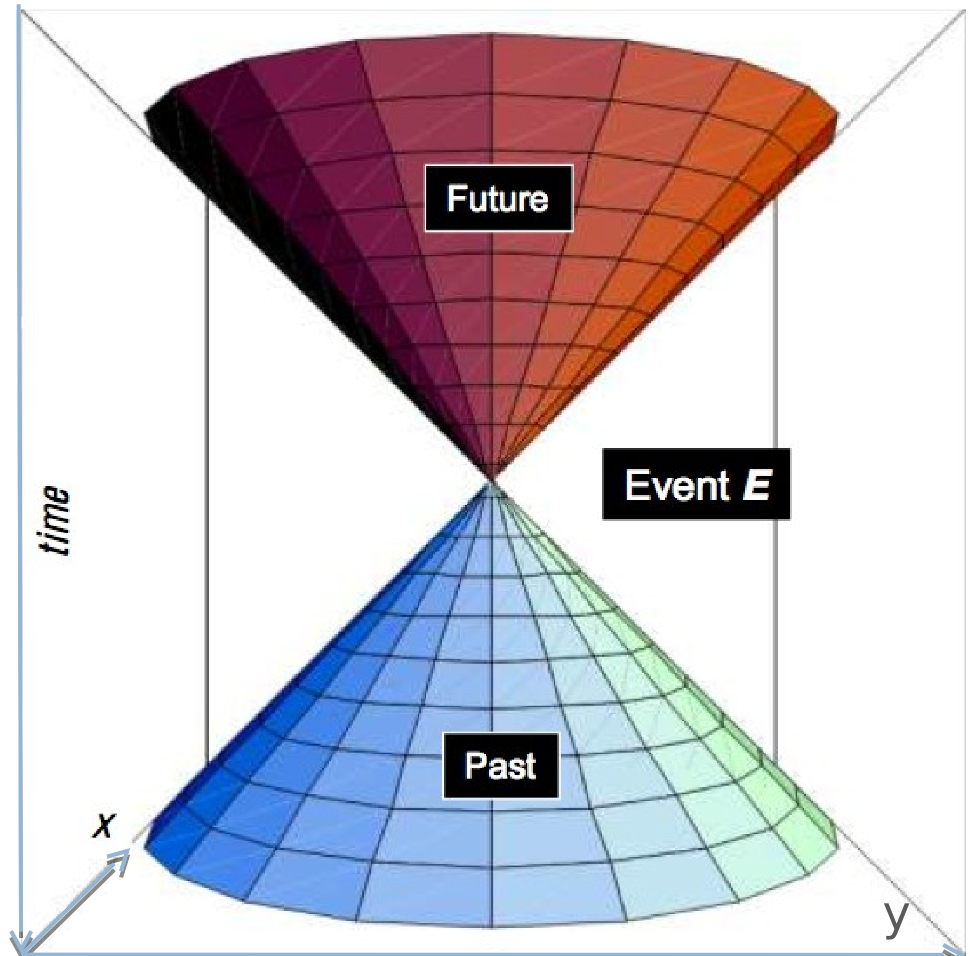
  1) If $a^2>0$, a^μ is called *timelike* because the a^0 component dominates the scalar product.

  2) If $a^2<0$, a^μ is called *spacelike* because the spatial components dominate a^2.

  3) If $a^2=0$, a^μ is called *lightlike* or null because, as with photons, the time and space components of a^μ cancel.

The light cone, revisited

- A set of points all connected to a single event E by lines moving at the speed of light is called the light cone.
- The set of points inside the light cone are timelike separated from E.
- The set of points outside the cone are spacelike separated from E.
- Points outside the cone cannot causally affect (or be affected by) the event E; signals from these points cannot make it to the event.



Past and future light cones for an event E, with z dimension suppressed, and units defined such that c=1.

Back to particle physics…

Why is relativity so prevalent and fundamental in this field?

SR in particle physics

- We will talk about relativistic kinematics –the physics of particle collisions and decays.
- In the context of what we have discussed so far, we start to think of particles as moving “observers”, and scientists as stationary observers.
  - The reference frame of particles is often called the **“particle rest frame**”, while the frame in which the scientist sits at rest, studying the particle, is called the **“lab frame”**.
- To begin, let’s *define* (not derive) the notions of relativistic energy, momentum, and the mass-energy relation. These should reduce to classical expressions when velocities are very low (classical limit).

We will be applying the algebra of 4-vectors to particle physics.

Relativistic momentum

- The relativistic momentum (a three-vector) of a particle is similar to the momentum you're familiar with, except for one of those factors of γ:

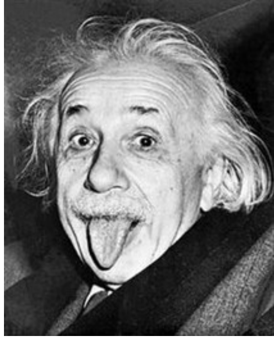
$$\vec{p} = \gamma m\vec{v} = \frac{m\vec{v}}{\sqrt{1 - v^2/c^2}}$$

- The relativistic momentum agrees with the more familiar expression in the so-called "classical regime" where v is a small fraction of c. In this case:

$$\vec{p} = m\vec{v}\left(1 + \frac{1}{2}\frac{v^2}{c^2} + \ldots\right) \approx m\vec{v}$$

(Taylor expansion)

Relativistic energy



- The relativistic energy (excluding particle interactions) is quite a bit different from the classical expression:

$$E = \gamma mc^2 = \frac{mc^2}{\sqrt{1 - v^2/c^2}}$$

- When the particle velocity v is much smaller than c, we can expand the denominator (Taylor expansion again) to get:

$$E = mc^2\left(1 + \frac{v^2}{2c^2} + \frac{3v^4}{8c^4} + \dots\right) = mc^2 + \frac{1}{2}mv^2 + \frac{3}{8}m\frac{v^4}{c^2} + \dots$$

- The second term here corresponds to the classical kinetic energy, while the leading term is a constant. (This is not a contradiction in the classical limit, because in classical mechanics, we can offset particle energies by arbitrary amounts.)
- The constant term, which survives even when v=0, is called the *rest* energy of the particle; it is Einstein's famous equation: $E_{\text{rest}} = mc^2$

Energy-momentum four-vector

- It is convenient to combine the relativistic energy and momentum into a single 4-vector called the *four-momentum.*

$$p^{\mu} = \left(\frac{E}{c}, \vec{p} \right) = (\gamma mc, \gamma m\vec{v})$$

- The four-momentum, denoted p^{μ} or just p, is defined by:
- The scalar product of the four-momentum with itself gives us an invariant that depends on the mass of the particle under study. Squaring p^{μ} yields the famous *relativistic energy-momentum relation* (also called the mass-shell formula):

$$p \cdot p = \frac{E^2}{c^2} - \vec{p}^2 = m^2 c^2$$

$$E = \sqrt{|\vec{p}|^2 c^2 + m^2 c^4}$$

The Lorentz-invariant quantity that results from squaring 4-momentum is called the invariant mass.

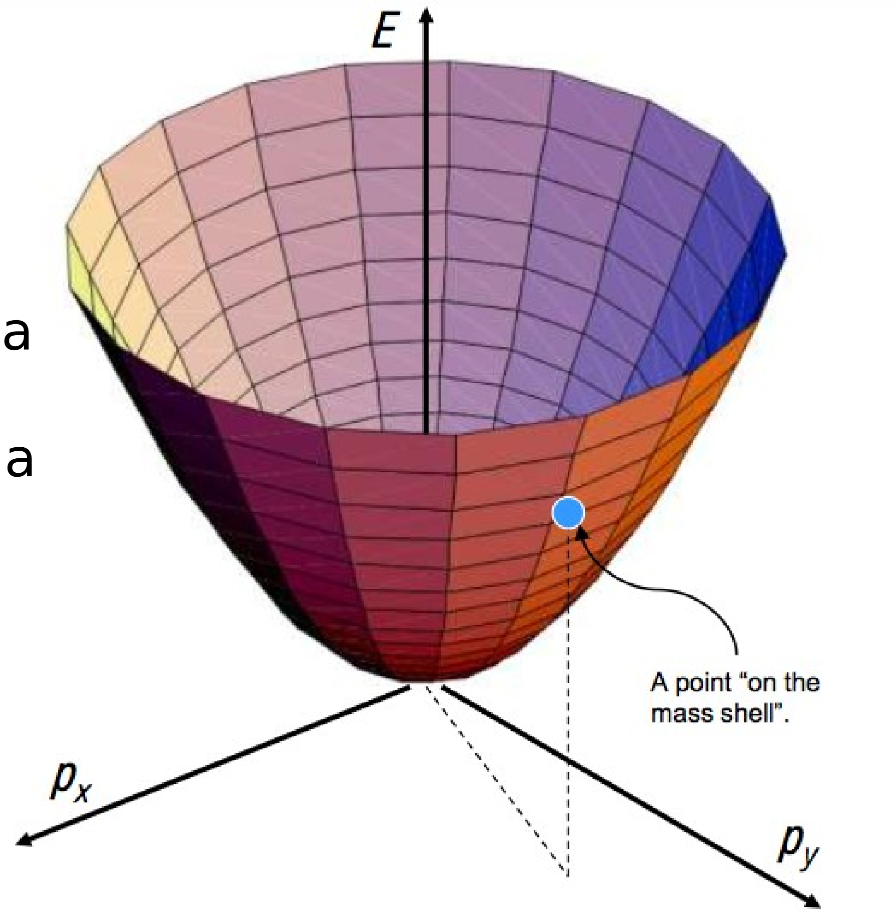
Classical vs. relativistic mass shell

- In *classical physics*, the mass-shell relation is quadratic in the momentum:

$$E = \frac{\vec{p}^2}{2m} = \frac{\left(p_x^{\ 2} + p_y^{\ 2} + p_z^{\ 2}\right)}{2m}$$

- This is called the mass-shell formula because if one plots E vs. p in two dimensions, the function looks like a parabolic shell(!).
- JARGON: Particles that obey the relativistic mass-shell relation are said to be "on mass shell":

$$E = \sqrt{\left|\vec{p}\right|^2 c^2 + m^2 c^4}$$



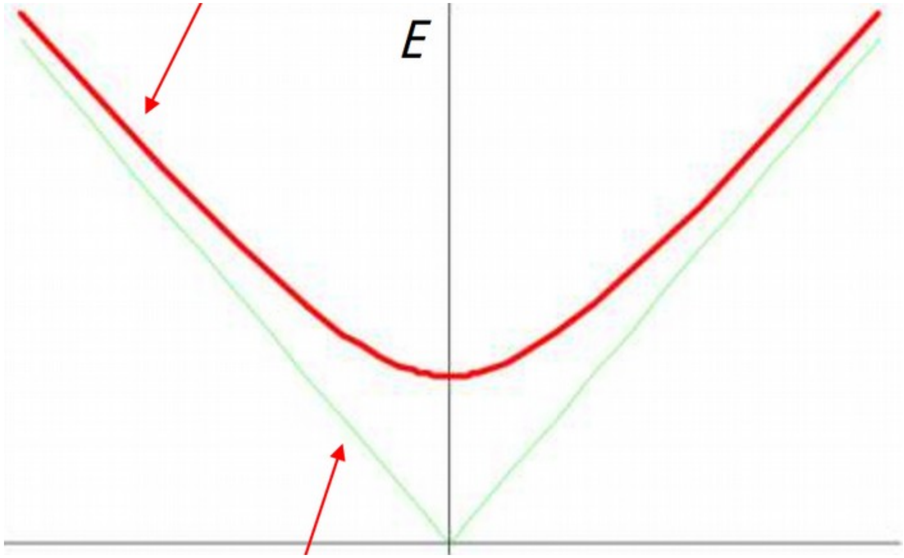
Classical mass shell relation for a 2D system.

Classical vs. relativistic mass shell

- The *relativistic* mass shell, due to the presence of the rest energy, looks like a hyperbola.
- Unlike classical mechanics, zero-mass particles are allowed if they travel at the speed of light.
- In the case of zero mass, the mass-shell relation reduces to:

$$E = |\vec{p}|c$$

Relativistic mass shell for 1D motion (m≠0).



Relativistic mass shell for 1D motion (m=0) (boundary of the light cone).

Collisions and kinematics

- Why have we introduced energy and momentum?
  - These quantities are *conserved* in any physical process (true in any inertial frame!).

- The cleanest application of these conservation laws in particle physics is to collisions.
- The collisions we will discuss are somewhat idealized; we essentially treat particles like billiard balls, ignoring external forces like gravity or electromagnetic interactions.
- Is this a good approximation? Well, if the collisions occur fast enough, we can ignore the effects of external interactions (these make the calculation much harder)!

Classical vs. relativistic collisions

- In classical collisions, recall the usual conservation laws:
  1) Mass is conserved;
  2) Momentum is conserved;
  3) Kinetic energy may or may not be conserved.
- The types of collisions that occur classically include:

  1) Sticky: kinetic energy decreases
  2) Explosive: kinetic energy increases
  3) Elastic: kinetic energy is conserved.

Classical vs. relativistic collisions

**Note: conservation of energy and momentum can be encompassed into conservation of four-momentum.*

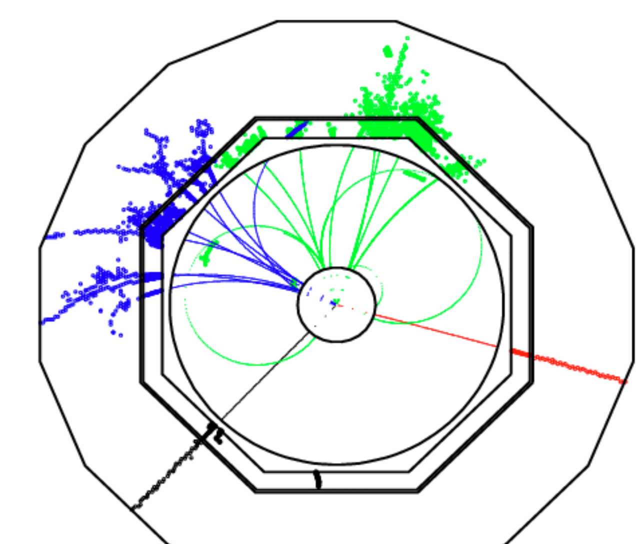
- In ~~classical~~ relativistic collisions, recall the usual conservation laws:
  1) ~~Mass~~ Relativistic energy is conserved;
  2) ~~Momentum~~ Relativistic momentum is conserved;
  3) Kinetic energy may or may not be conserved.
- The types of collisions that occur classically include:

  1) Sticky: kinetic energy decreases, rest energy and mass increase.
  2) Explosive: kinetic energy increases, rest energy and mass decrease.
  3) Elastic: kinetic energy is conserved, rest energy and mass are conserved too.

Inelastic collisions

- There is a difference in interpretation between classical and relativistic inelastic collisions.
- In the classical case, inelastic collisions mean that kinetic energy is converted into "internal energy" in the system (e.g., heat).
- In special relativity, we say that the kinetic energy goes into rest energy.
- Is there a contradiction? No, because the energy-mass relation $E=mc^2$ tells us that all "internal" forms of energy are manifested in the rest energy of an object. (In other words, hot objects weigh more than cold objects. But this is not a measurable effect even on the atomic scale!)

Mass-energy equivalence



Summary

- Lorentz boosts to and from a moving reference frame:

$$\begin{aligned} x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \\ t' &= \gamma(t - vx/c^2) \end{aligned} \quad , \quad \begin{aligned} x &= \gamma(x' + vt') \\ y &= y' \\ z &= z' \\ t &= \gamma(t' + vx'/c^2) \end{aligned} \quad , \quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \geq 1$$

- Relativistic momentum and energy:

$$\vec{p} = \gamma m \vec{v},$$ relativistic momentum

$$E = \gamma mc^2,$$ relativistic energy

$$E_{\text{rest}} = mc^2,$$ rest energy

$$T = E - E_{\text{rest}} = (\gamma - 1)mc^2,$$ relativistic kinetic energy

$$E^2 = |\vec{p}|^2 c^2 + m^2 c^4,$$ mass-shell relation