Particle Physics

Columbia Science Honors Program

Week 4: Quantum Mechanics

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Course Policies

- Attendance:
  - I will take attendance during class.
  - Up to four excused absences (two with notes from parent/guardian)
  - Send notifications of all absences to shpattendance@columbia.edu
- Valid excuses:
  - Illness, family emergency, tests or athletic/academic competitions, mass transit breakdowns
- Invalid excuses: sleeping in, missing the train
- Please no cell phones.
- Ask questions :)

Lecture Materials

- https://twiki.nevis.columbia.edu/twiki/bin/view/Main/ScienceHonorsProgram

Last week…

Relativistic mechanics

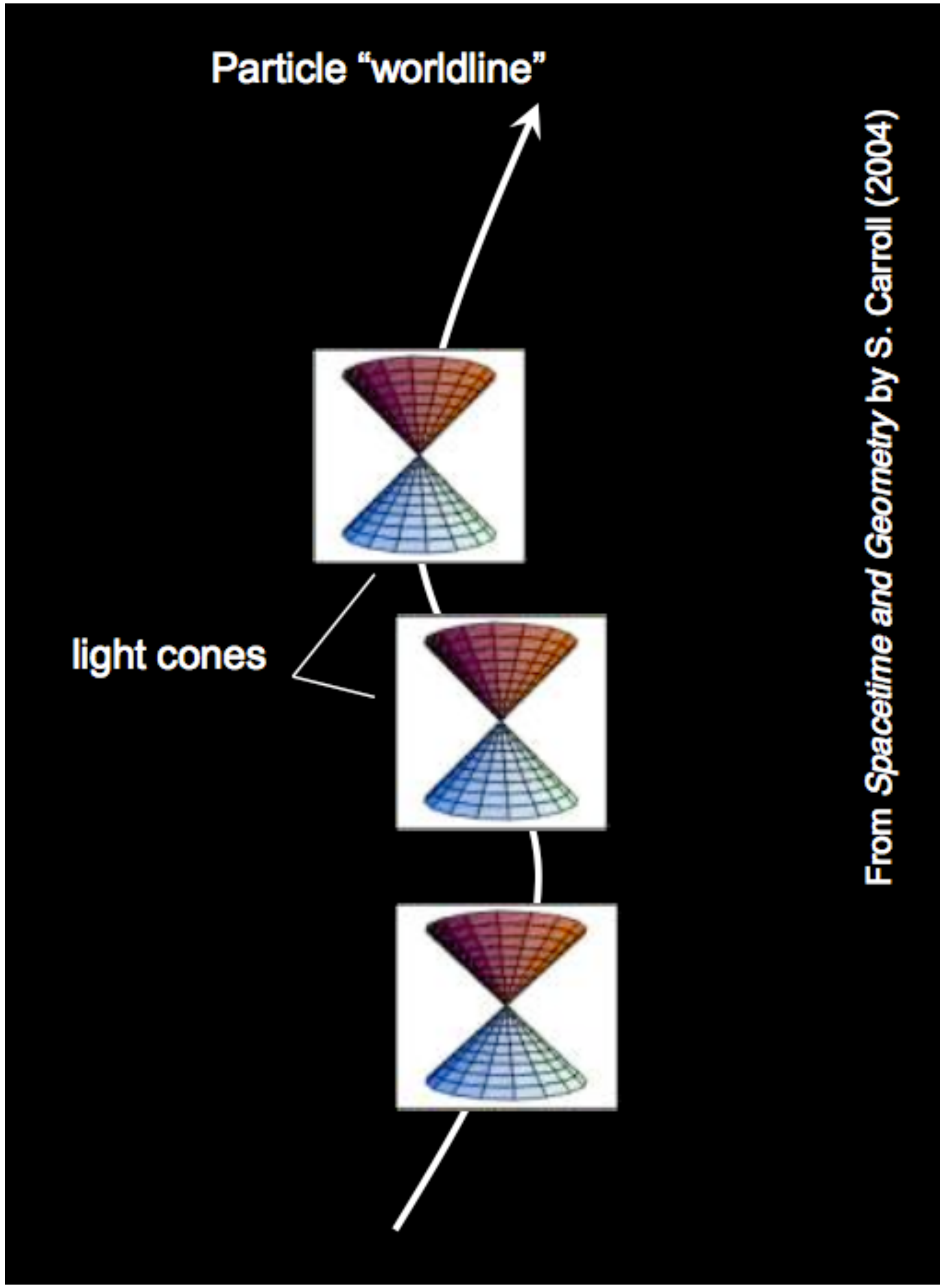
What's wrong with classical mechanics?

- We will see that classical mechanics is only valid in the limiting case where $v \rightarrow 0$, or $v << c$.
  - This is generally the case for everyday observables.
- However, this is not the case with particles traveling close to the speed of light.
  - In that case, classical mechanics fails to describe their behavior.
  - To properly describe particle kinematics, and particle dynamics, we need relativistic mechanics.

The notion of spacetime

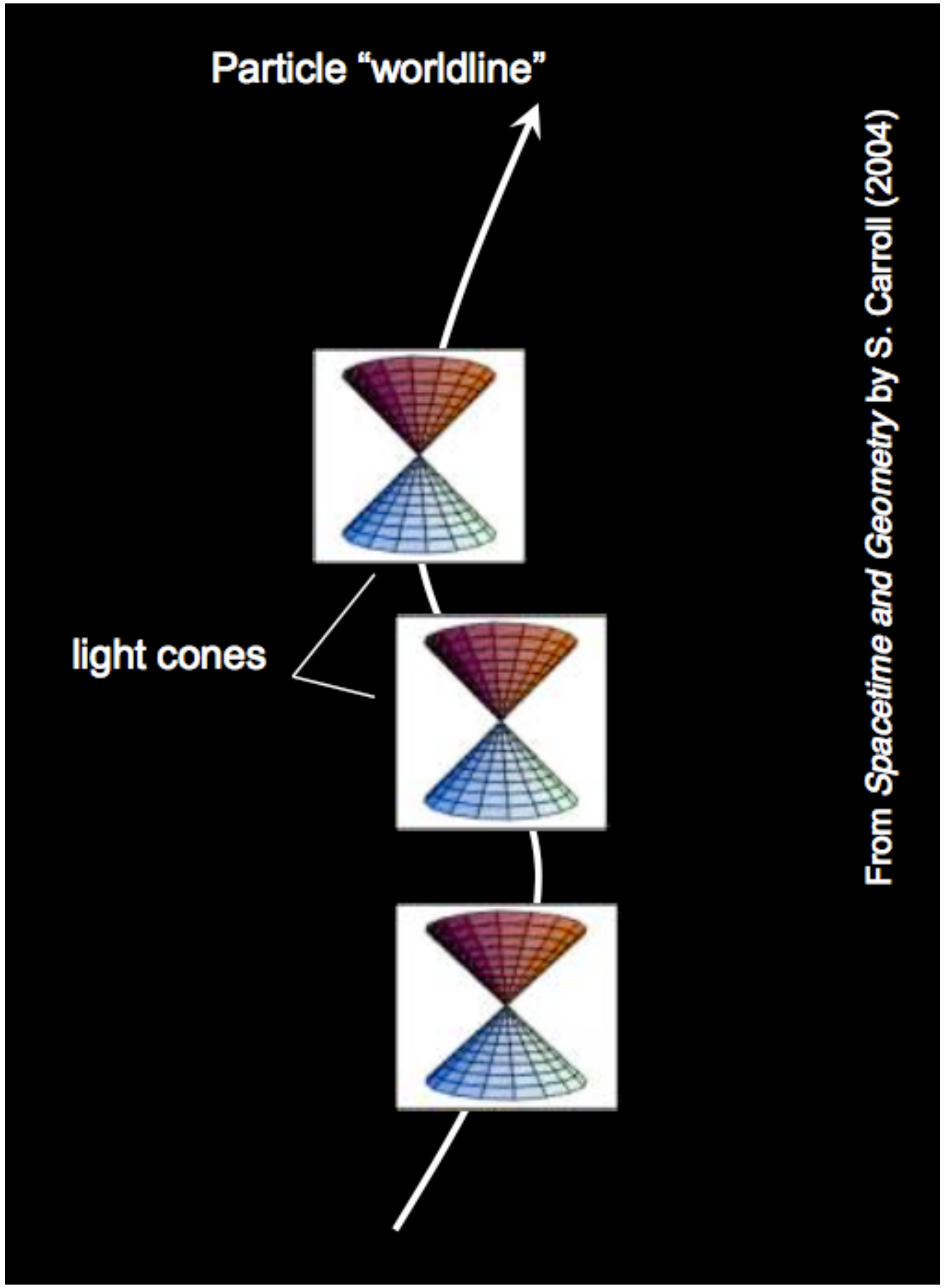
Spacetime in Special Relativity

- Time is local.
- Observers may not agree that two events occur at the same time.
- There is no absolute notion of all space at a moment in time.
- The speed of light is constant, and cannot be surpassed.
- Every event "exists" within a set of allowed trajectories (light cone).



Basic concepts

- **Event:** something that occurs at a specified point in space, at a specified time.
- **Observer**: someone who witnesses and can describe events (also known as a "frame of reference")
  - An observer describes events by using **"standard" clocks and rulers** which are at rest with respect to him/her.



Postulates of special relativity

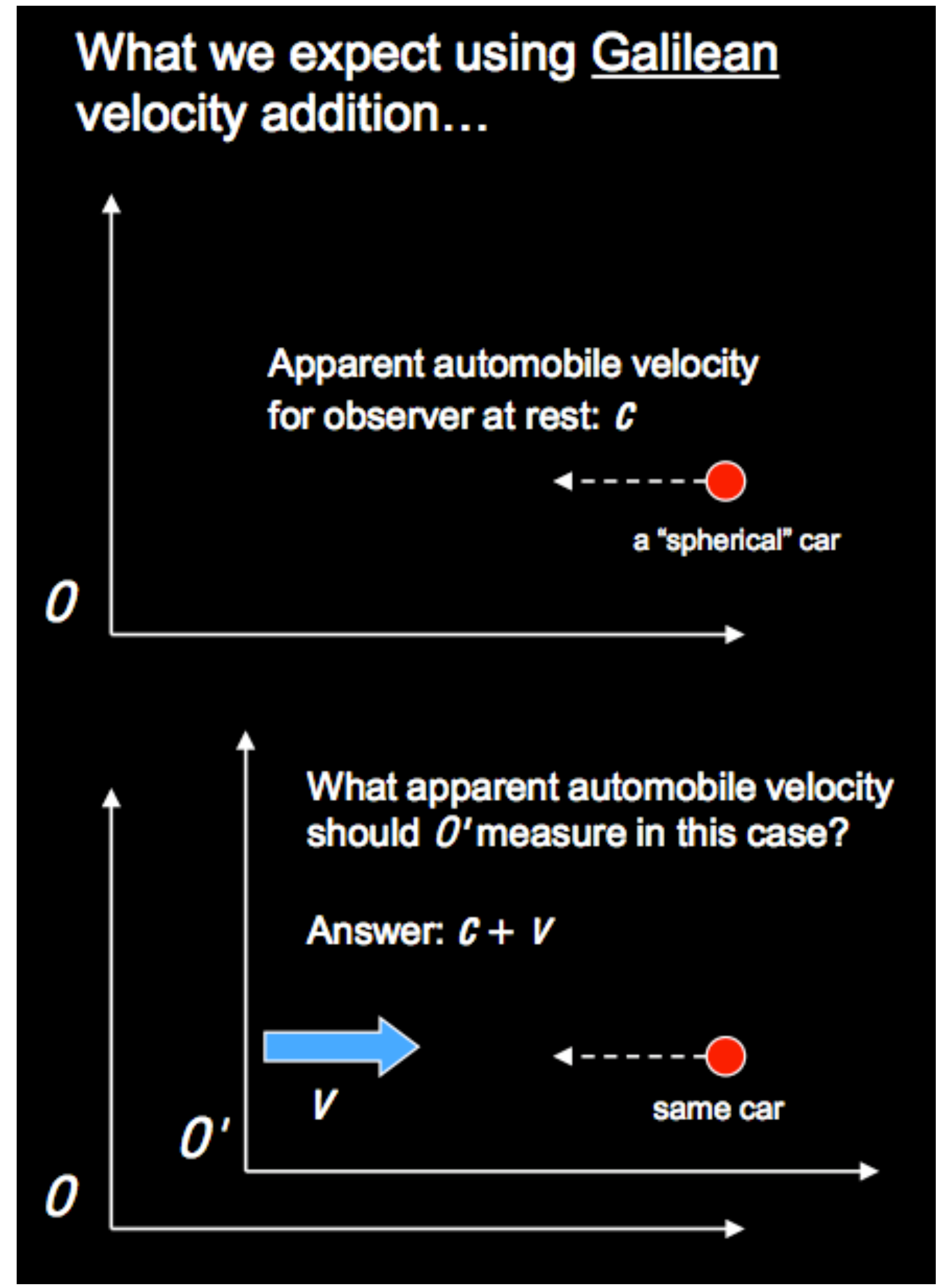
- In 1905, A. Einstein published two papers on special relativity, as well as a paper on the photoelectric effect (Nobel Prize 1921) and Brownian motion (the physics of particles suspended in a fluid).

- All of Einstein's conclusions in special relativity were based on only two simple postulates:

1. The laws of physics are the same in all inertial reference frames (old idea, dates to Galileo).
2. All inertial observers measure the same speed c for light in a vacuum, independent of the motion of the light source.

Postulates of special relativity

- The constancy of the speed of light is counter-intuitive, because this is not how "ordinary" objects behave.
- Example: imagine observing an oncoming car that moves at speed *c*.
  - We expect a moving observer to measure a different value for *c* than a stationary one.
  - According to SR, however, we always measure *c* for light, regardless of our motion!



Implications of the postulates

- Einstein developed a series of "thought experiments" that illustrate the interesting consequences of the universality of *c*. These can be summarized as:

1. The illusion of simultaneity
2. Time dilation
3. Lorentz (length) contraction
4. Velocity addition (not really a thought experiment)

As we go through Einstein's examples, keep in mind that these results may seem a little counterintuitive.
You have to get rid of your Newtonian way of thinking!

(The relativity of) Simultaneity

- An observer O calls two events simultaneous if they occur at the same time in his / her coordinates.
- Interestingly, if the two events do not occur at the same *position* in frame O, then they will not appear simultaneous to a moving observer O'.
- In other words, events that are simultaneous in one inertial system are not necessarily simultaneous in others.
- **Simultaneity is not an absolute concept, but one that depends on the state of motion of the observer.**
- Again, this follows from the fact that *c* is the same in all inertial frames…

(The relativity of) Simultaneity

- A demonstration: Einstein's thought experiment.

"A boxcar moves with uniform velocity and two lightning bolts strike the ends of the boxcar, leaving marks on the boxcar and ground. The marks on the boxcar are labeled A' and B'; on the ground, A and B. The events recorded by the observers are the light signals from the lightning bolts. The two light signals reach observer O at the same time."

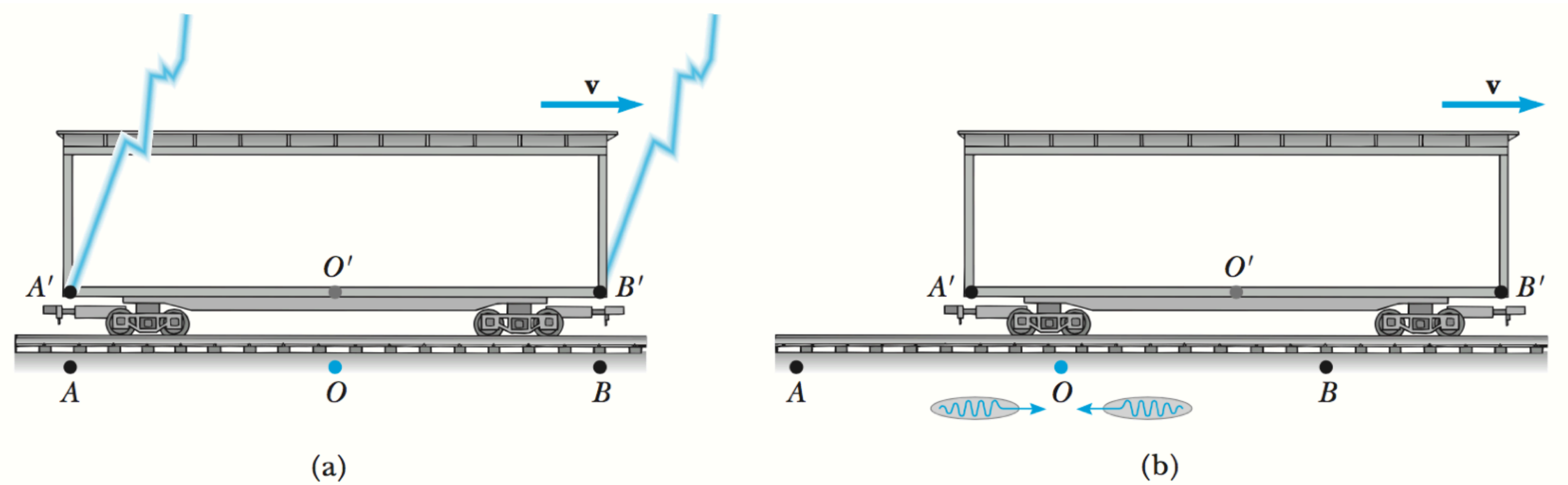
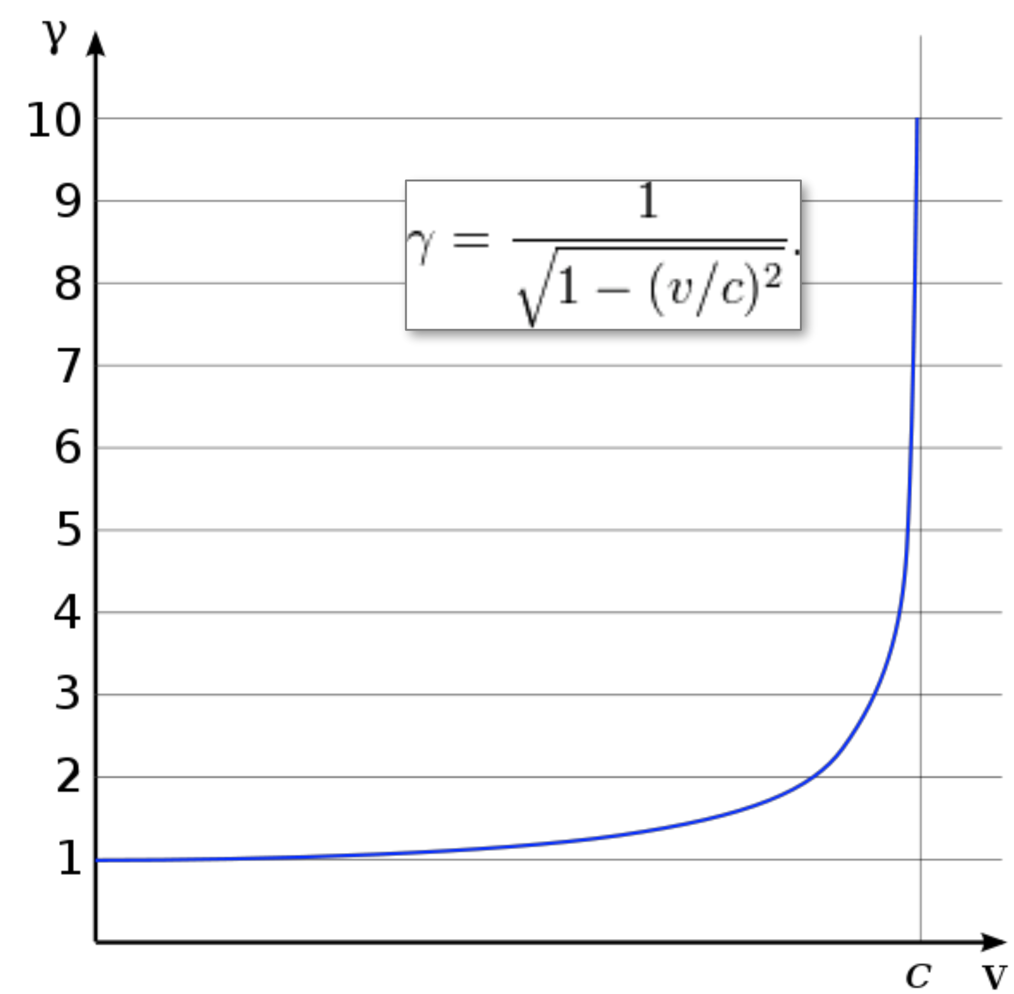


Figure 1.9 Two lightning bolts strike the ends of a moving boxcar. (a) The events appear to be simultaneous to the stationary observer at *O*, who is midway between *A* and *B*. (b) The events do not appear to be simultaneous to the observer at *O'*, who claims that the front of the train is struck *before* the rear.

Time dilation

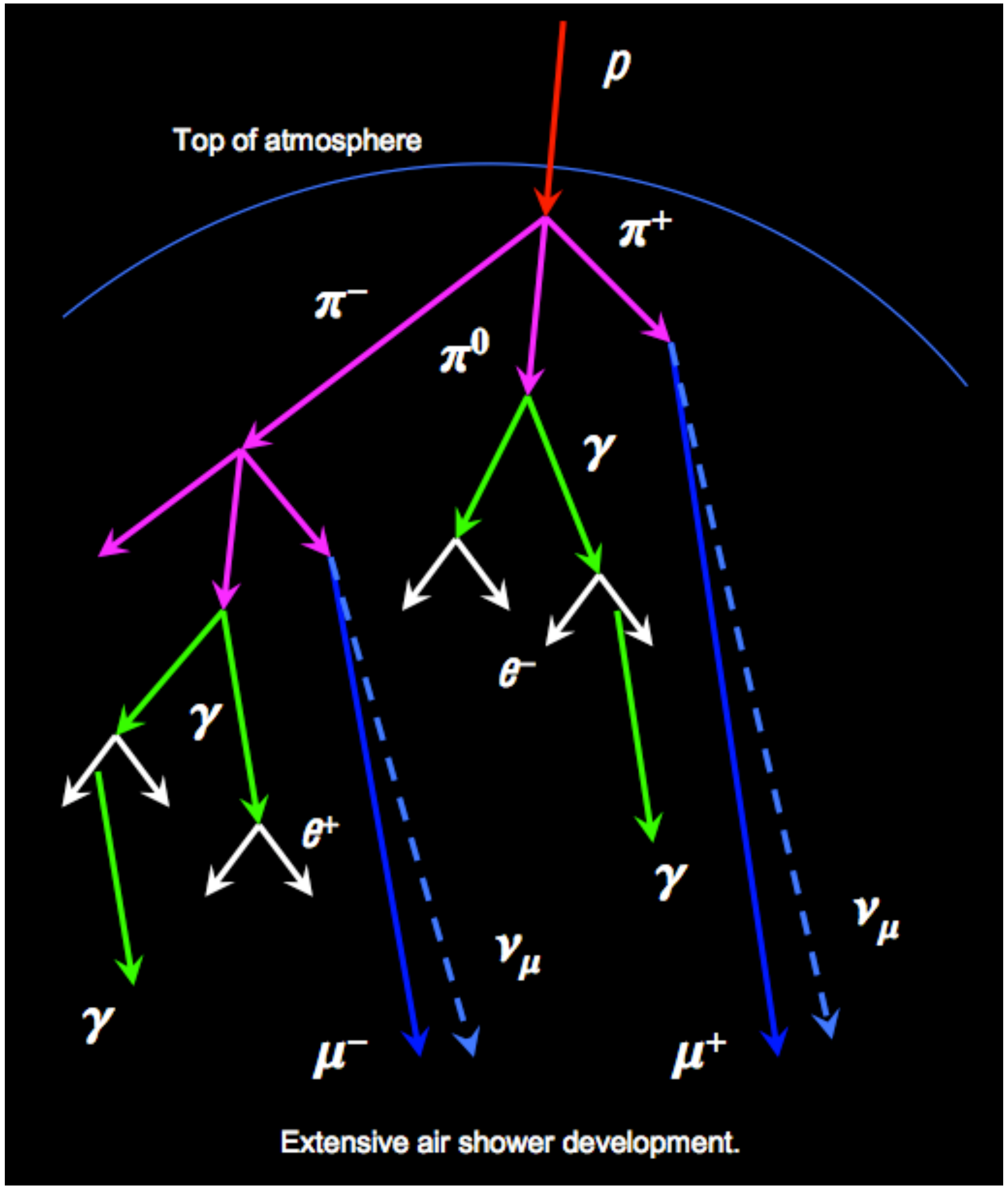
- Time dilation reflects the fact that observers in different inertial frames always measure different time intervals between a pair of events.
- Specifically, an observer O at rest will measure a longer time between a pair of events than an observer O' in motion, i.e., moving clocks tick more slowly than stationary clocks!
- The amount by which the observer at rest sees the time interval "dilated" with respect to the measurement by O' is given by the factor called Lorentz factor γ:



$$\Delta t = \frac{\Delta t'}{\sqrt{1 - v^2 / c^2}} = \gamma \Delta t'$$

Time dilation in practice

- Recall our mention of cosmic ray air showers…
- Relativistic nuclei strike the atmosphere, causing a huge cascade of high energy decay products.
  - Many of these are detected at Earth's surface.
- However, most of them (like π's and μ's) are very unstable and short-lived.
- How do they make it to Earth's surface?



Time dilation in practice

Naively:

- The mean lifetime of the muon (in its rest frame) is 2.2 microseconds.
- Most air shower muons are generated high in the atmosphere (~8 km altitude).
- If they travel at 99.9% of the speed of light *c*, should they make it to Earth from that altitude?

$$\begin{aligned}\text{Muon range} &= (\text{lifetime}) \times (\text{speed}) \\ &= (2.2 \times 10^{-6}\ \text{s}) \times (0.999c) \\ &\approx 660\ \text{m}\end{aligned}$$

This suggests that muons should not be able to make it to Earth's surface. But we detect them. Where did the calculation go wrong?

Time dilation in practice

Accounting for relativity:

- In the lab (the stationary frame), the muon's lifetime undergoes time dilation (a muon's clock ticks slower…).
- Therefore, we have an effective lifetime to deal with:

$$\text{Muon range} = \gamma \times (\text{lifetime}) \times (\text{speed})$$
$$= \left(\frac{1}{\sqrt{1-(0.999c/c)^2}} \right) \times (2.2\times 10^{-6}\ \text{s}) \times (0.999c)$$
$$\approx 14.7\ \text{km}$$

So the muon can certainly make it to the ground, on average, when we account for relativistic effects.

Transformation between reference frames

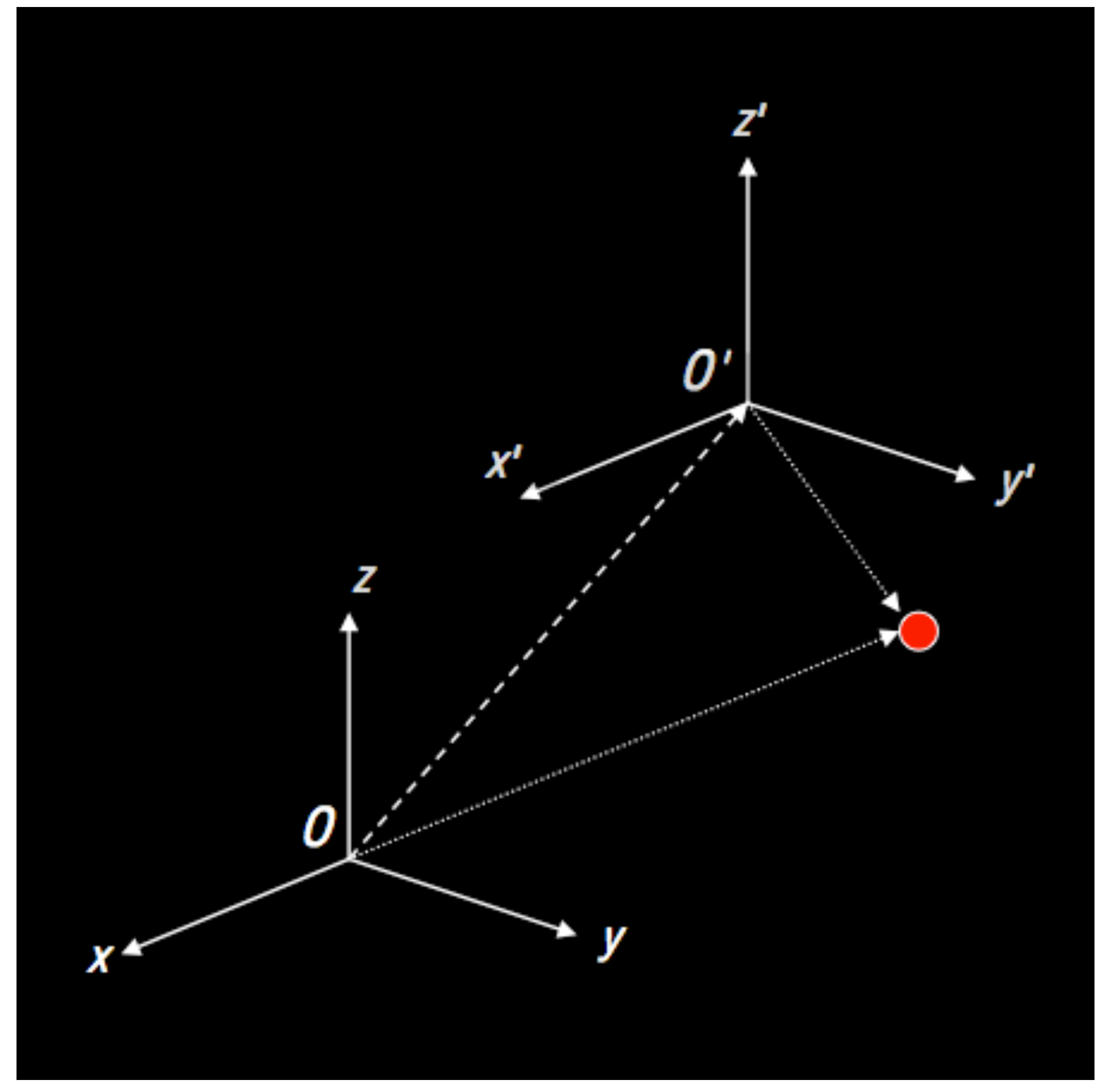
- Using the postulates of Special Relativity, we can start to work out how to transform coordinates between different inertial observers.
  - What is a transformation? It's a mathematical operation that takes us from one inertial observer's coordinate system into another's.
- The set of possible transformations between inertial reference frames are called the **Lorentz Transformations**.
- They form a group (in the mathematical sense of "group theory").
- The possible Lorentz Transformations:
  - Translations
  - Rotations
  - Boosts

Translations (fixed displacements)

- In fixed translations, the two observers have different origins, but don't move with respect to each other.
- In this case, the observers' clocks differ by a constant b_0 and their positions differ by a constant vector b:

$$\vec{x}' = \vec{x} - \vec{b}$$

$$t' = t - b_0$$

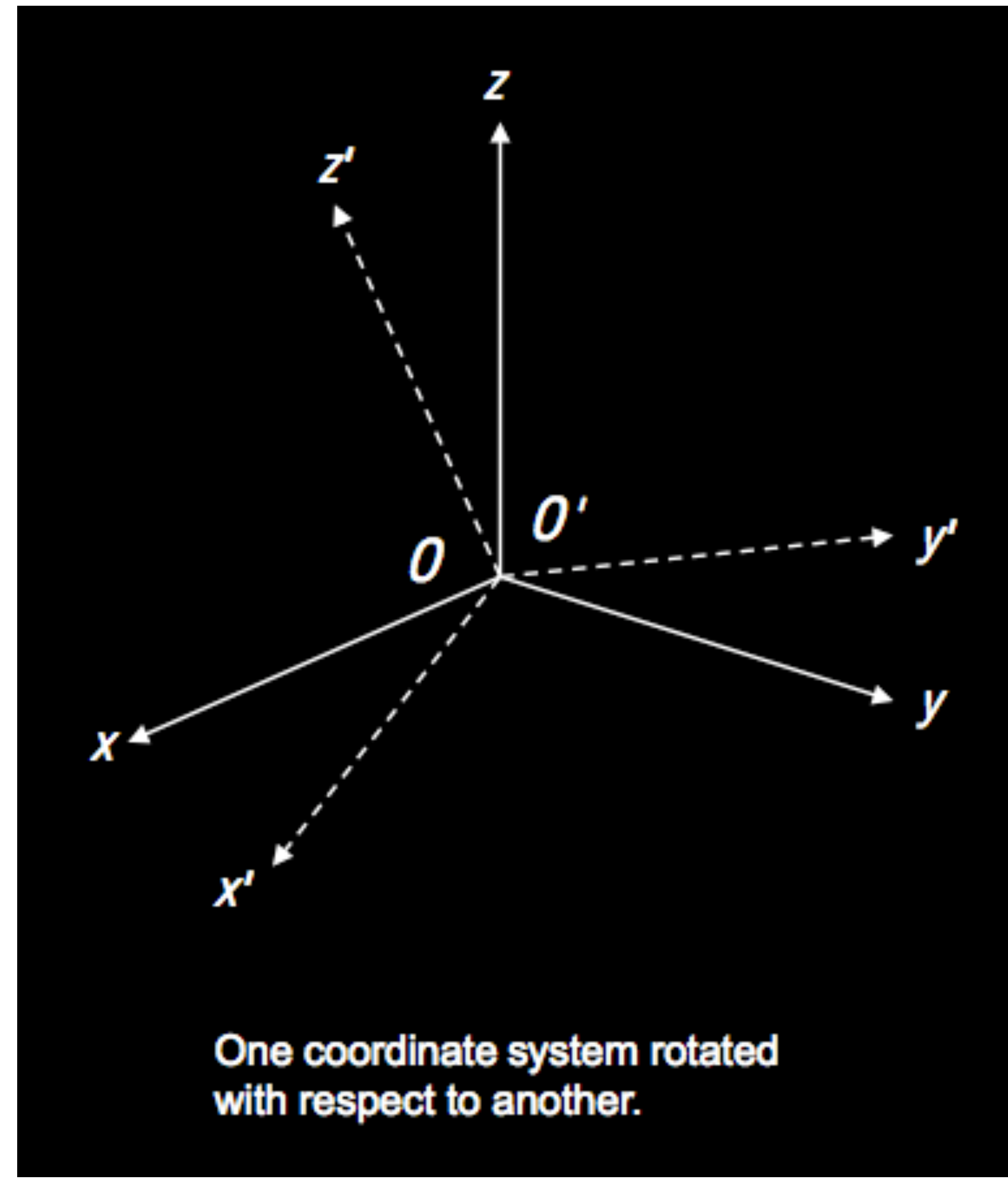


Rotations (fixed)

- In fixed rotations, the two observers have a common origin and don't move with respect to each other.
- In this case, the observers' coordinates are rotated with respect to each other.
- The spatial transformation can be accomplished with a rotation matrix; measured times are the same:

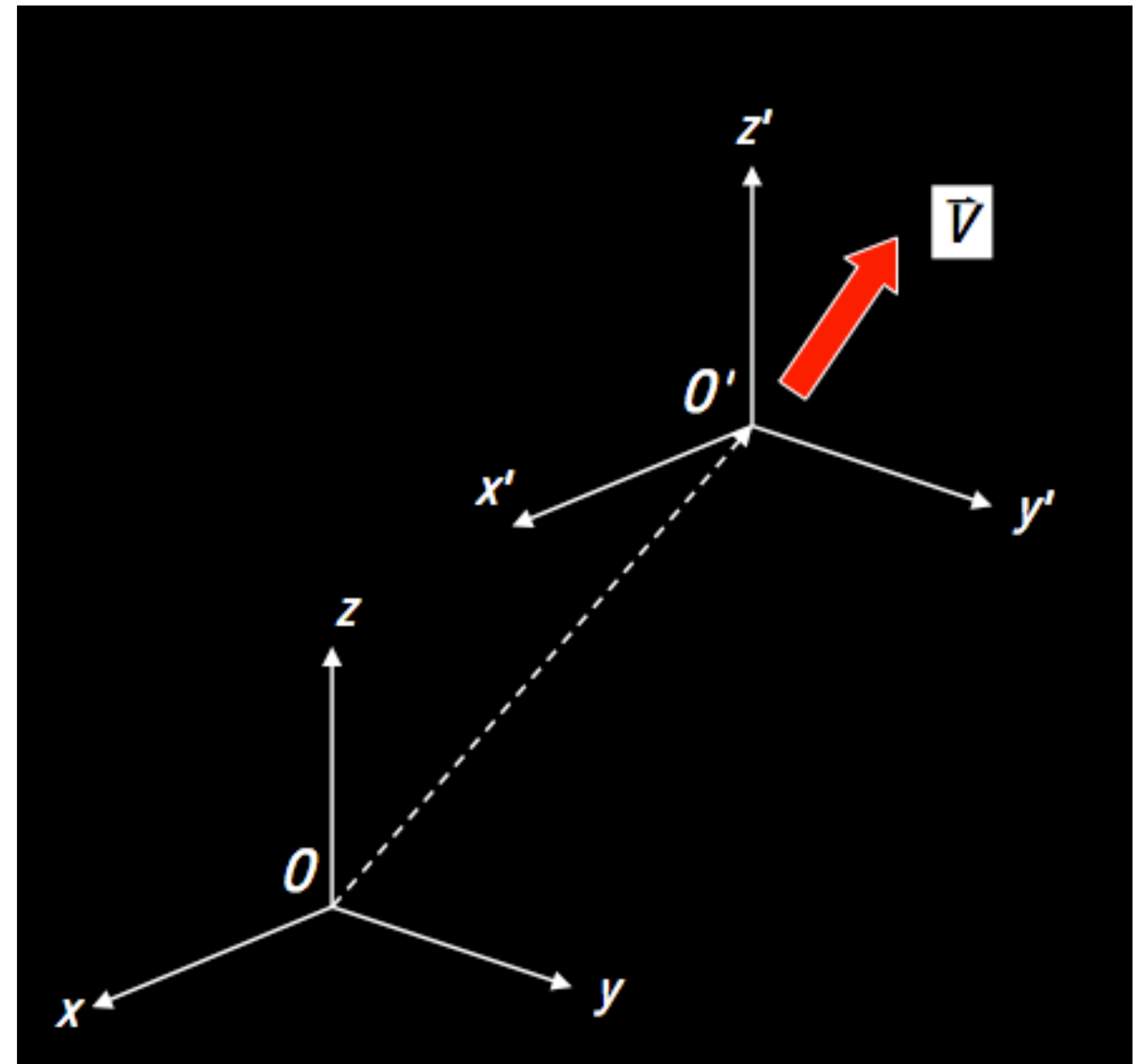
$$\vec{x}' = \overleftrightarrow{R} \cdot \vec{x}$$

$$t' = t$$



Boosts

- In boosts, the two frame axes are aligned, but the frames move at constant velocity with respect to each other.
- The origins are chosen here to coincide at time t=0 in both frames.
- The fact that the observers' coordinates are not fixed relative to each other makes boosts more complex than translations and rotations.
- It is in boosts that the constancy of the speed of light plays a big role.



Observers' axes are aligned, but one moves at constant velocity *v* with respect to the other.

Boosts: Galileo vs Lorentz

- Suppose we have two observers O and O'. O is at rest, and O' moves along the x direction with constant velocity v.
- According to Galileo, the transformation between the coordinates of O and O' is pretty simple.
- According to Lorentz and Einstein, we get complicated expressions with many factors of c involved: the so-called Lorentz transformations.
- If an event occurs at position (x,y,z) and time t for observer O, what are the space-time coordinates (x',y',z') and t' measured by O'?

Galileo

$$x' = x - vt$$
$$y' = y$$
$$z' = z$$
$$t' = t$$

Lorentz

$$x' = \gamma(x - vt)$$
$$y' = y$$
$$z' = z$$
$$t' = \gamma(t - vx/c^2)$$

$$, \quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \geq 1$$

Note the Lorentz factor γ in the Lorentz boosts.

Lorentz (length) contraction

- Suppose a moving observer O' puts a rigid "meter" stick along the x' axis: one end is at x'=0 and the other at $x' = L' = L_0$.
- Now an observer O at rest measures the length of the stick at time t=0, when the origins of O and O' are aligned. What will O measure for the length?
- Using the first boost equation $x' = \gamma(x - vt)$ at time t=0, it looks like the lengths are related by:

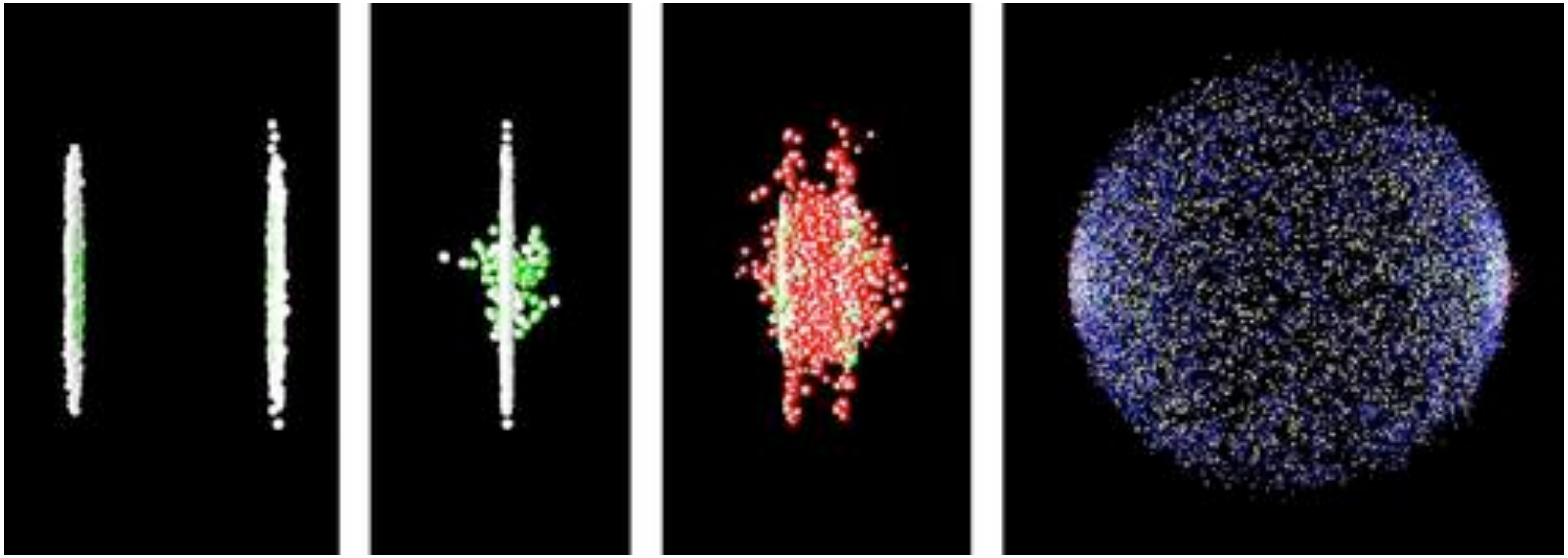
$$\text{moving} \quad L' = \gamma\, L \quad \text{at rest}$$

$$L = L' / \gamma$$

This is the **Lorentz contraction**: if an object has length L_0 when it is at rest, then when it moves with speed v in a direction parallel to its length, an observer at rest will measure its length as the shorter value L_0/γ.

Lorentz Contraction

- An example of Lorentz contraction in the case of collisions of two gold nuclei at the RHIC collider at Brookhaven Lab on Long Island:



- In typical collisions (200 GeV), nuclei have a Lorentz factor of O(200).

Velocity addition

- Finally, let's briefly derive the rule for addition of relativistic velocities (we will need to use the boost equations…)
- Suppose a particle is moving in the x direction at speed u' with respect to observer O'. What is its speed u with respect to O?
  - Since the particle travels a distance $\Delta x=\gamma(\Delta x'+v\Delta t')$–an "inverse" boost –in time $\Delta t=\gamma\,(\Delta t'+(v/c^2)\,\Delta x')$, the velocity in frame O is:

$$\frac{\Delta x}{\Delta t}=\frac{\Delta x'+v\Delta t'}{\Delta t'+(v/c^2)\Delta x'}=\frac{(\Delta x'/\Delta t')+v}{1+(v/c^2)(\Delta x'/\Delta t')}$$

where v is the relative velocity of the two inertial frames.

  - Since $u=\Delta x/\Delta t$ and $u'=\Delta x'/\Delta t'$, we get the addition rule:

$$u=\frac{u'+v}{1+(u'v/c^2)};\quad \text{compare to}\quad u=u'+v$$

Four-vector notation

- This is a way to simplify notation for all we've talked about so far.
- Soon after Einstein published his papers on Special Relativity, Minkowski noticed that **regarding t and (x,y,z) as simply four coordinates in a 4-D space ("space-time")** really simplified many calculations.
- In this spirit, we can introduce a position-time four-vector x^μ, where μ=0,1,2,3, as follows:

$$x^0 = ct, \quad x^1 = x, \quad x^2 = y, \quad x^3 = z$$

Lorentz boosts in four-vector notation

- In terms of the 4-vector x^μ, a Lorentz boost along the x^1 (that is, the x) direction looks like:

$$\begin{aligned} x^{0\prime} &= \gamma(x^0 - \beta x^1) \\ x^{1\prime} &= \gamma(x^1 - \beta x^0) \\ x^{2\prime} &= x^2 \\ x^{3\prime} &= x^3 \end{aligned}, \quad \text{where } \beta = v/c$$

- As an exercise, you can show that the above equations recover the Lorentz boosts we discussed earlier.
- FYI, this set of equations also has a very nice and useful matrix form.

Lorentz boosts in matrix form

- Using 4-vectors, we can write the Lorentz boost transformation as a matrix equation:

$$\begin{pmatrix} x^{0\prime} \\ x^{1\prime} \\ x^{2\prime} \\ x^{3\prime} \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

- Looks very similar to the 3-D rotation!
- Mathematically, boosts and rotations are actually very close "cousins". We can understand this connection using the ideas of group theory.

Invariant quantities

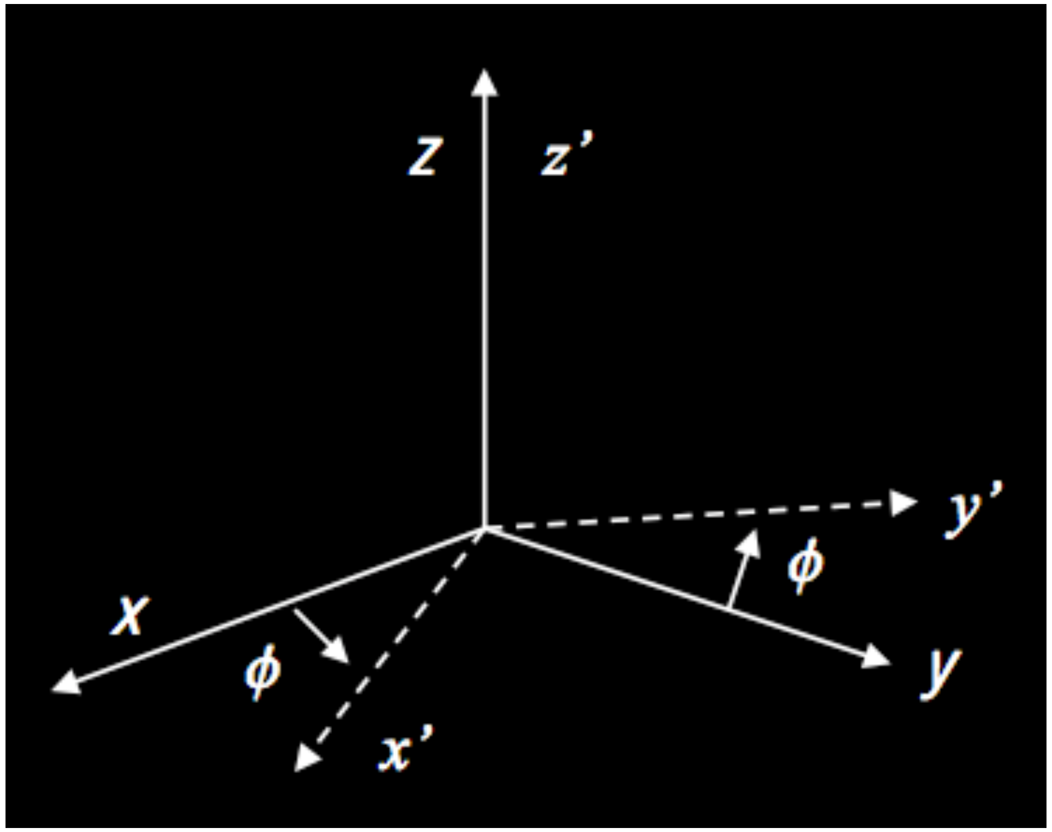
- The utility of 4-vectors comes in when we start to talk about invariant quantities.
- **Definition: a quantity is called invariant if it has the same value in any inertial system.**
- Recall: the laws of physics are always the same in any inertial coordinate system (this is the definition of an inertial observer).
  - Therefore, these laws are invariants, in a sense.
- The identification of invariants in a system is often the best way to understand its physical behavior.

Example of invariant quantity

- Think of a 3-vector (x,y,z). An example of an invariant is its square magnitude: $r^2=x^2+y^2+z^2$, whose value does not change under coordinate rotations.
- Consider a rotation about the z-axis:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{aligned} r'^2 &= x'^2 + y'^2 + z'^2 \\ &= (x\cos\varphi + y\sin\varphi)^2 + (-x\sin\varphi + y\cos\varphi)^2 + z^2 \\ &= (\cos^2\varphi + \sin^2\varphi)(x^2 + y^2) + z^2 \\ &= r^2 \end{aligned}$$



4-vector scalar product

- The quantity Δs^2, given by:

$$\Delta s^2 = x^0 x^0 - x^1 x^1 - x^2 x^2 - x^3 x^3 = x^0 x^0 - \vec{x} \cdot \vec{x}$$
$$= (ct)^2 - x^2$$

is called the scalar product of x^μ with itself.

- It has the same value in any coordinate system (just like any scalar).
  - This spacetime interval is often called the proper length.
- To denote the scalar product of two arbitrary 4-vectors a^μ and b^μ, it is convenient to drop the Greek index and just write:

$$a \cdot b = a^0 b^0 - \vec{a} \cdot \vec{b}$$

- In this case, the 4-vectors *a* and *b* are distinguished from their spatial 3-vector components, by the little arrow overbar.

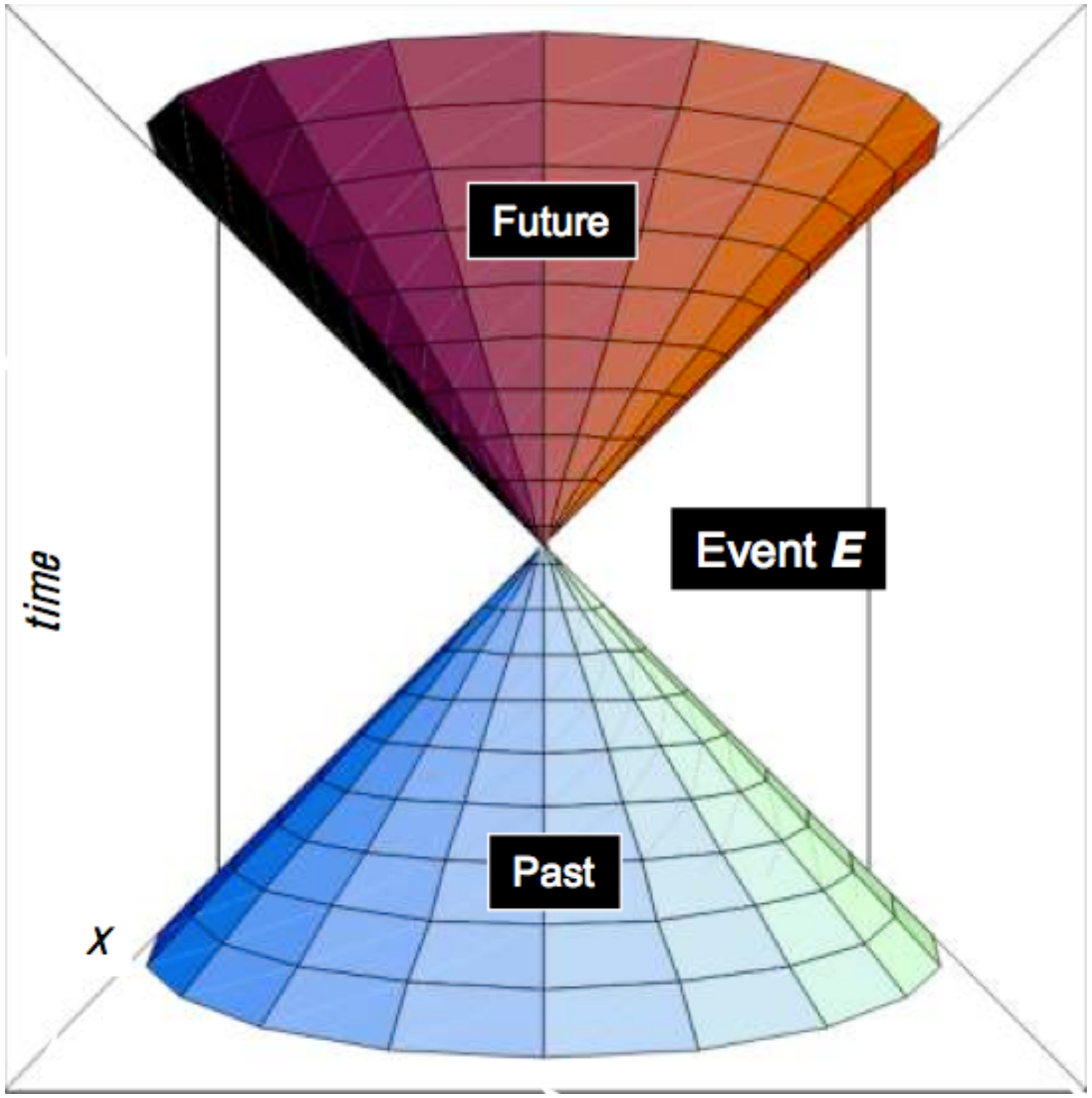
4-vector scalar product

- Terminology: any arbitrary 4-vector a^μ can be classified by the sign of its scalar product a^2:

1. If $a^2>0$, a^μ is called *time-like* because the time component dominates the scalar product.
2. If $a^2<0$, a^μ is called *space-like* because the spatial components dominate a^2.
3. If $a^2 = 0$, a^μ is called *light-like* or null because, as with photons, the time and space components of a^μ cancel.

The light cone, revisited

- A set of points all connected to a single event E by lines moving at the speed of light is called the **light cone**.
- The set of points inside the light cone are time-like separated from E.
- The set of points outside the cone are space-like separated from E.
- Points outside the cone cannot casually affect (or be affected by) the event E.
    - Signal from these points cannot make it to the event.



Past and future light cones for an event E, with z dimension suppressed, and units defined such that c=1.

Back to particle physics…

- Why is relativity so prevalent and fundamental in this field?

SR in particle physics

- We will talk about relativistic kinematics - the physics of particle collisions and decays.
- In the context of what we have discussed so far, we start to think of particles as moving "observers", and scientists as stationary observers.
  - The reference frame of particles is often called the "particle rest frame", while the frame in which the scientist sits at rest, studying the particle, is called the "lab frame".
- To begin, let's *define* (not derive) the notions of relativistic energy, momentum, and the mass-energy relation.
  - These should reduce to classical expressions when velocities are very low (classical limit).

We will be applying the algebra of 4-vectors to particle physics.

Relativistic momentum

- The relativistic momentum (a three-vector) of a particle is similar to the momentum you're familiar with, except for one of those factors of γ:

$$\vec{p} = \gamma m\vec{v} = \frac{m\vec{v}}{\sqrt{1 - v^2/c^2}}$$

- The relativistic momentum agrees with the more familiar expression in the so-called "classical regime" where *v* is a small fraction of *c*.
- In this case:

$$\vec{p} = m\vec{v}\left(1 + \frac{1}{2}\frac{v^2}{c^2} + \ldots\right) \approx m\vec{v}$$

(Taylor expansion)

Relativistic energy

- The relativistic energy (excluding particle interactions) is quite a bit different from the classical expression:

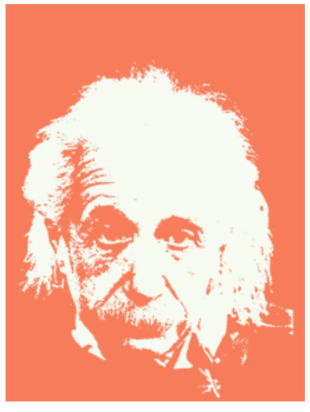
$$E = \gamma m c^2 = \frac{mc^2}{\sqrt{1 - v^2/c^2}}$$

- When the particle velocity v is much smaller than c, we can expand the denominator to get:

$$E = mc^2\left(1 + \frac{v^2}{2c^2} + \frac{3v^4}{8c^4} + \ldots\right) = mc^2 + \frac{1}{2}mv^2 + \frac{3}{8}m\frac{v^4}{c^2} + \ldots$$

- The second term here corresponds to the classical kinetic energy, while the leading term is a constant.
- This is not a contradiction in the classical limit, because in classical mechanics we can offset particle energies by arbitrary amounts.
- The constant term is called the rest energy of the particle and it is Einstein's famous equation:

$$E_{rest} = mc^2$$



Energy-momentum four-vector

- It is convenient to combine the relativistic energy and momentum into a single 4-vector called the four-momentum.

$$p^{\mu} = \left(\frac{E}{c}, \vec{p} \right) = (\gamma mc, \gamma m\vec{v})$$

- The four-momentum, denoted p^{μ} or just *p*, is defined by:

The scalar product of the four-momentum with itself gives us an invariant that depends on the mass of the particle under study.

- Squaring p^{μ} yields the famous relativistic energy-momentum relation (also called the mass-shell formula):

$$p \cdot p = \frac{E^2}{c^2} - \vec{p}^2 = m^2 c^2$$

$$E = \sqrt{|\vec{p}|^2 c^2 + m^2 c^4}$$

The Lorentz-invariant quantity that results from squaring 4-momentum is called the invariant mass.

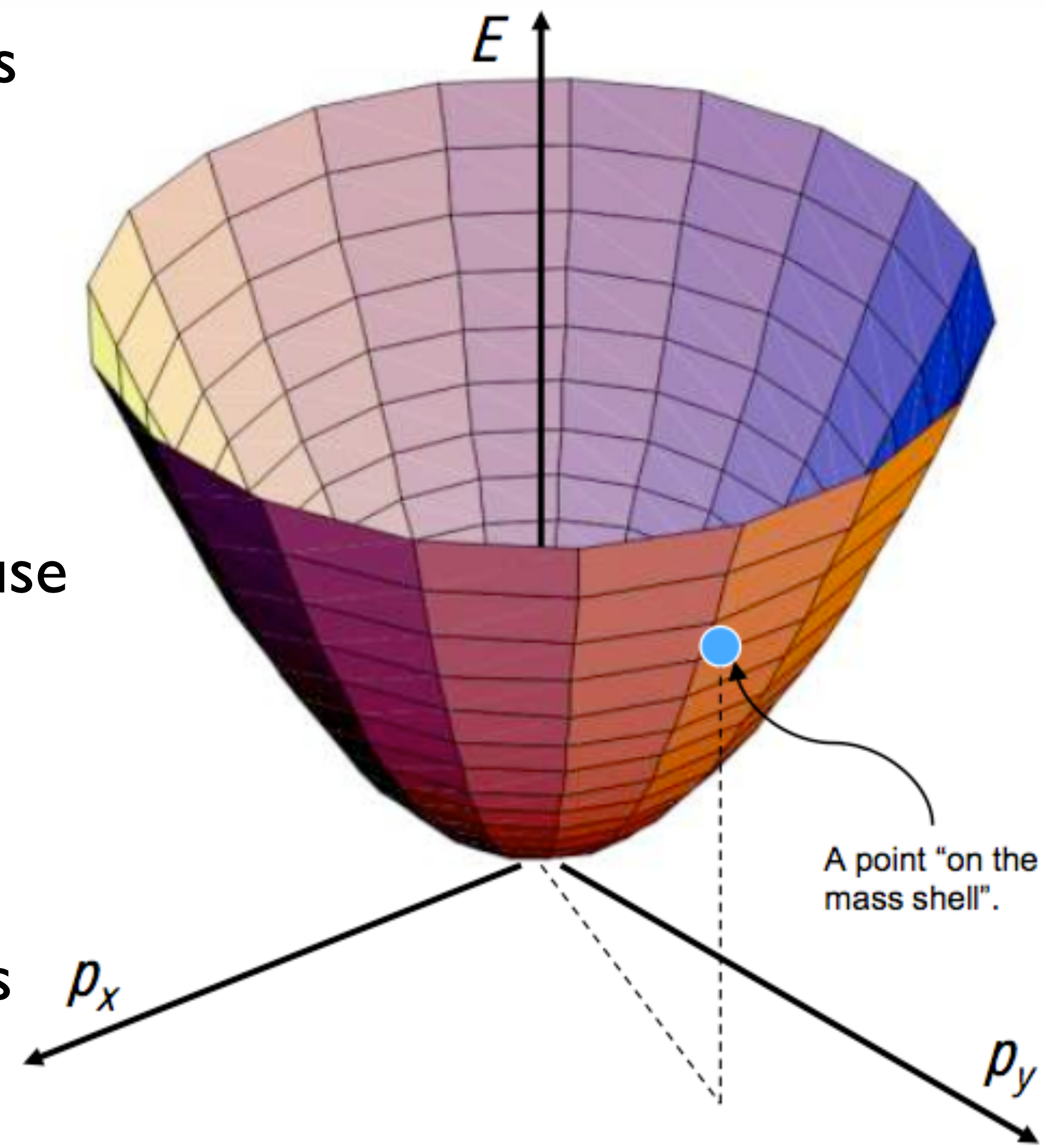
Classical vs. relativistic mass shell

- In *classical physics*, the mass-shell relation is quadratic in the momentum:

$$E = \frac{\vec{p}^2}{2m} = \frac{\left(p_x{}^2 + p_y{}^2 + p_z{}^2\right)}{2m}$$

- This is called the mass-shell formula because if one plots E vs p in two dimensions, the function looks like a parabolic shell.
- Jargon: particles that obey the relativistic mass-shell relation are said to be "on mass shell":

$$E = \sqrt{|\vec{p}|^2 c^2 + m^2 c^4}$$



Classical mass shell relation for a 2D system.

Classical vs. relativistic mass shell

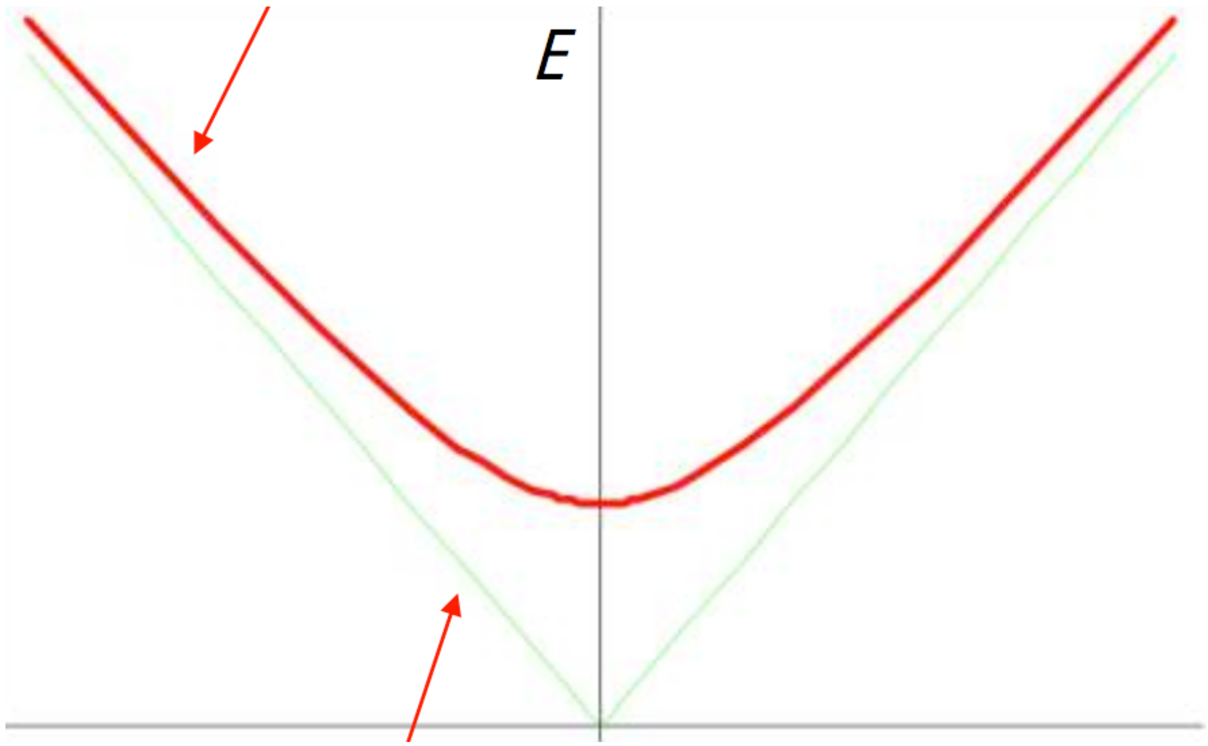
- The *relativistic* mass shell, due to the presence of the rest energy, looks like a hyperbola.

- Unlike classical mechanics, zero-mass particles are allowed if they travel at the speed of light.

- In the case of zero mass, the mass-shell relation reduces to:

$$E = |\vec{p}|c$$

I Relativistic mass shell for 1D motion ($m \neq 0$).



Relativistic mass shell for 1D motion ($m=0$) (boundary of the light cone).

Collisions and kinematics

- Why have we introduced energy and momentum?
    - **These quantities are conserved in any physical process (true in any inertial frame!).**
- The cleanest application of these conservation laws in particle physics is to collisions.
- The collisions we will discuss are somewhat idealized; we essentially treat particles like billiard balls, ignoring external forces like gravity or electromagnetic interactions.
- Is this a good approximation? Well, if the collisions occur fast enough, we can ignore the effects of external interactions (these make the calculation much harder!).

Classical vs. relativistic collisions

- In classical mechanics, recall the usual conservation laws:

1. Mass is conserved;
2. Momentum is conserved;
3. Kinetic energy may or may not be conserved.

Classical vs. relativistic collisions

- In ~~classical~~ relativistic mechanics, recall the usual conservation laws:

1. ~~Mass~~ is conserved; Relativistic energy
2. ~~Momentum~~ is conserved; Relativistic momentum
3. Kinetic energy may or may not be conserved.

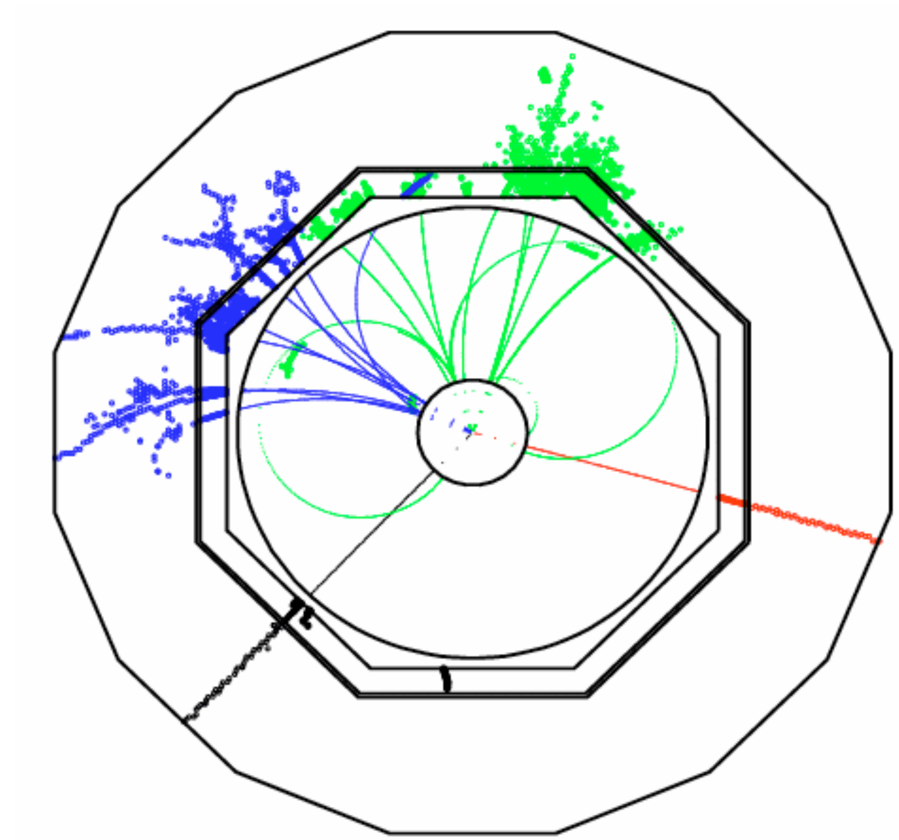
Note: conservation of energy and momentum can be encompassed into conservation of four-momentum.

Inelastic collisions

- There is a difference in interpretation between classical and relativistic inelastic collisions.
- In the classical case, inelastic collisions mean that kinetic energy is converted into "internal energy" in the system (e.g., heat).
- In special relativity, we say that the kinetic energy goes into rest energy.

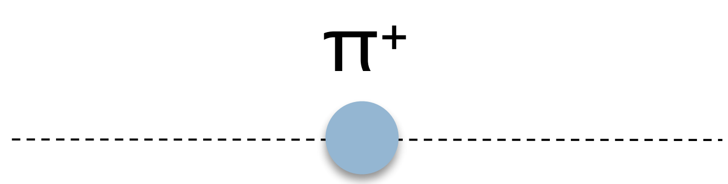
- Is there a contradiction?
    - No, because the energy-mass relation $E=mc^2$, tells us that all "internal" forms of energy are manifested in the rest energy of an object.
    - In other words, hot objects weigh more than cold objects. But this is not a measurable effect even on the atomic scale!

Mass-energy equivalence

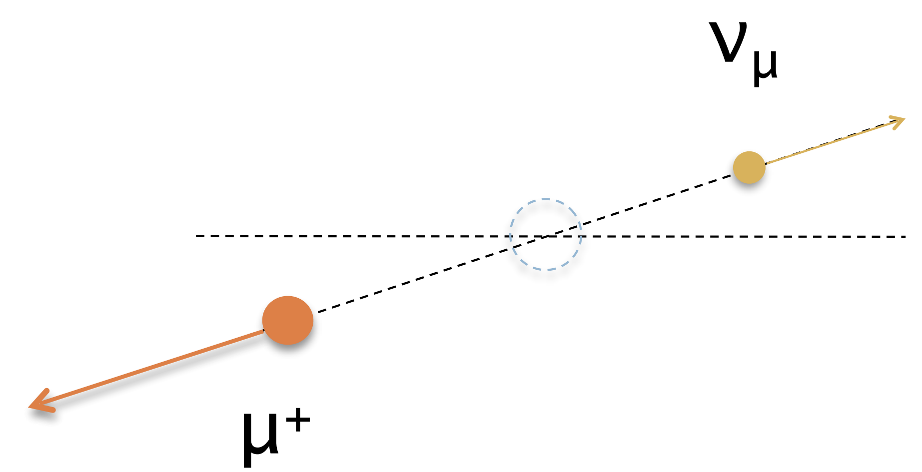


SR in particle physics

- Consider the decay $\pi^+ \to \mu^+ \nu^\mu$:



Before



After

Summary

- **Lorentz boosts** to and from a moving reference frame:

$$\begin{aligned} x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \\ t' &= \gamma(t - vx/c^2) \end{aligned} \quad , \quad \begin{aligned} x &= \gamma(x' + vt') \\ y &= y' \\ z &= z' \\ t &= \gamma(t' + vx'/c^2) \end{aligned} \quad , \quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \geq 1$$

- Relativistic momentum and energy:

$\vec{p} = \gamma m \vec{v},$	relativistic momentum
$E = \gamma mc^2,$	relativistic energy
$E_{\text{rest}} = mc^2,$	rest energy
$T = E - E_{\text{rest}} = (\gamma - 1)mc^2,$	relativistic kinetic energy
$E^2 = \lvert\vec{p}\rvert^2 c^2 + m^2 c^4,$	mass-shell relation

Schedule

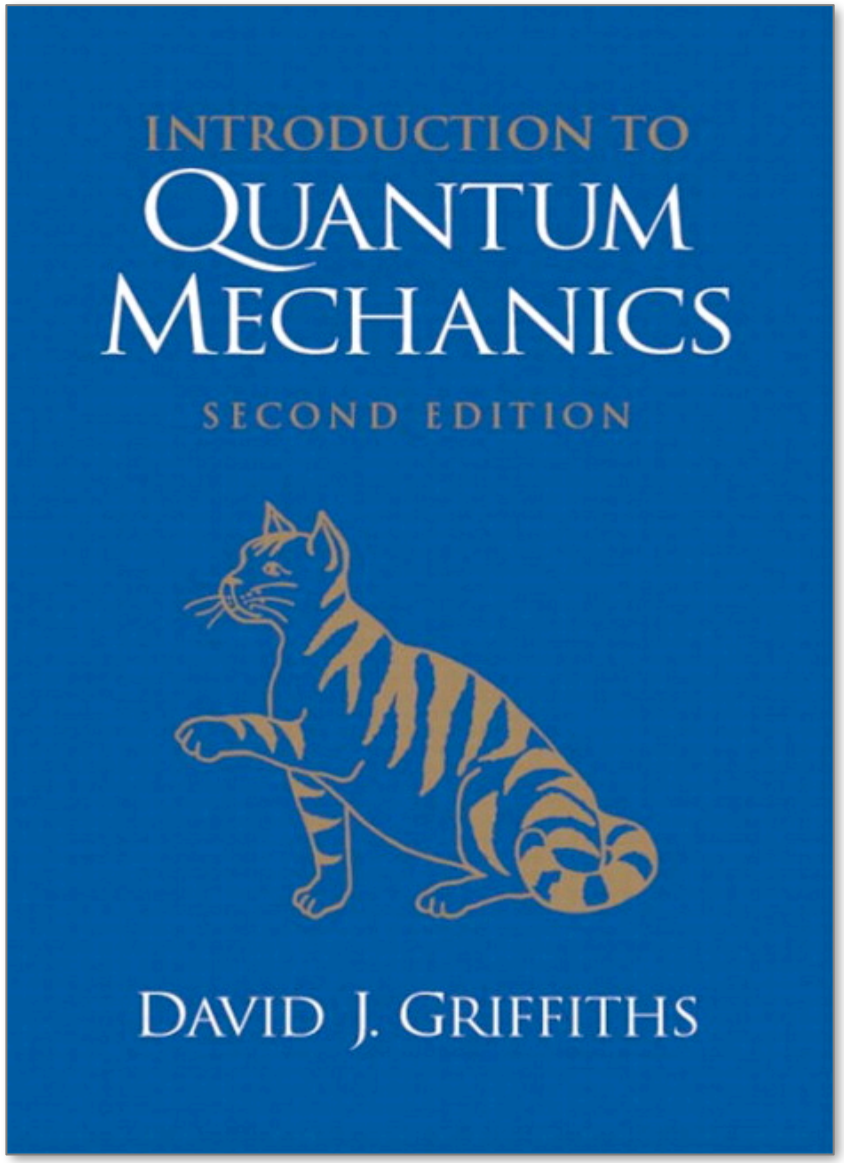
1. ~~Introduction~~
2. ~~History of Particle Physics~~
3. ~~Special Relativity~~
4. Quantum Mechanics
5. Experimental Methods
6. The Standard Model - Overview
7. The Standard Model - Limitations
8. Neutrino Theory
9. Neutrino Experiment
10. LHC and Experiments
11. The Higgs Boson and Beyond
12. Particle Cosmology

Quantum Mechanics

Particles and Waves

Topics for Today

- **Quantum phenomena:**
  - Quantization: how Nature comes in discrete packets
  - Particulate waves and wavelike particles
  - Understanding quantum phenomena in terms of waves.
  - The Schrödinger Equation
- **Interpreting quantum mechanics (QM):**
  - The probabilistic interpretation of quantum mechanics
  - The Uncertainty Principle and the limits of observation.
- **Understanding the Uncertainty Principle:**
  - Building up wave packets from sinusoids.
  - Why the Uncertainty Principle is a natural property of waves.



Recommended reading

What is Quantum Mechanics (QM)?

- QM is the study of physics at very small scales - specifically, when the energies and momenta of a system are of the order of Planck's constant:

$$\hbar = h/2\pi \cong 6.6\times10^{-16} \mathrm{eV.s}$$

- On the quantum level, "particles" exhibit a number of non-classical behaviors:

1. Discretization (quantization) of energy, momentum charge, spin, etc. Most quantities are multiples of e and/or h.
2. Particles can exhibit wavelike effects: interference, diffraction, …
3. Systems can exist in a superposition of states.

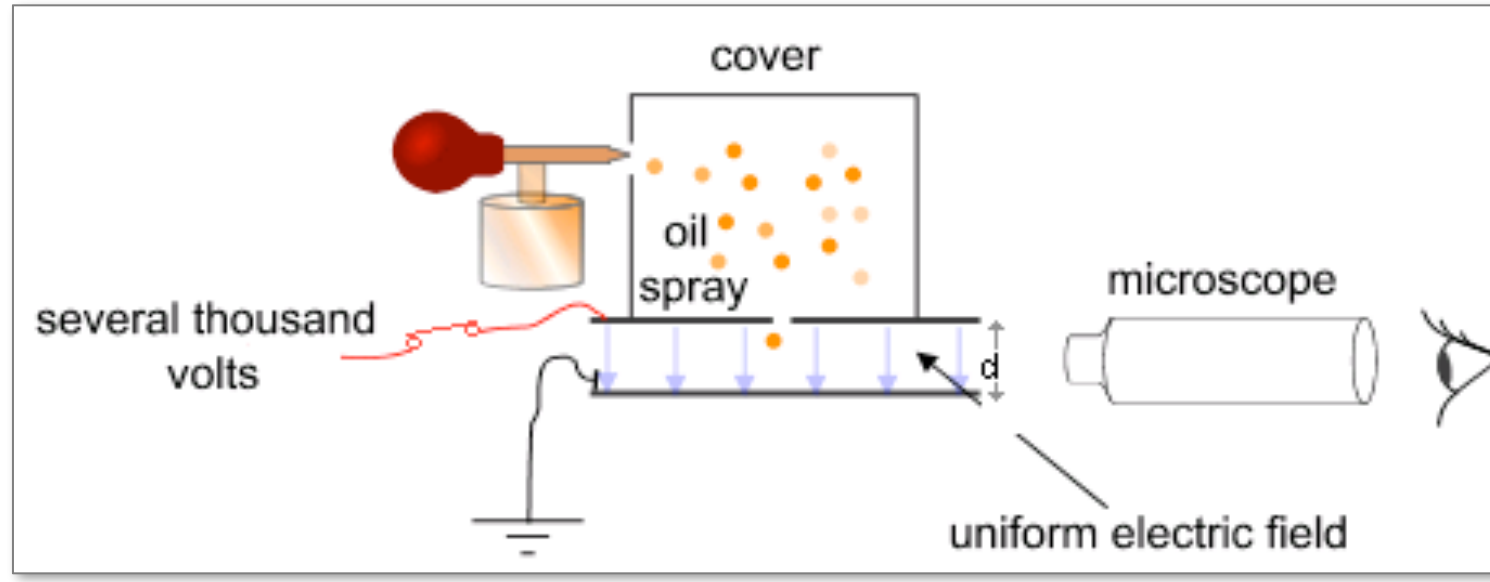
Quantization of electric charge

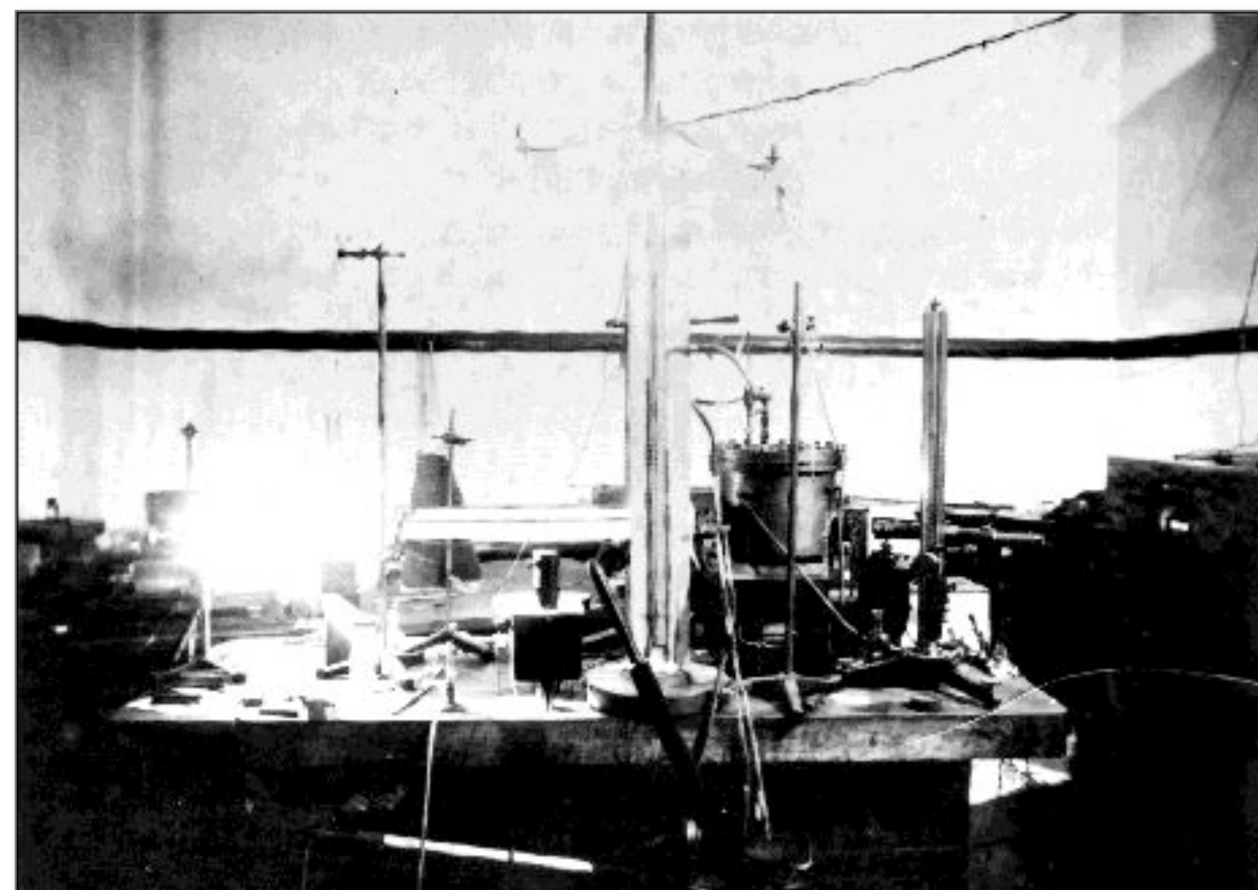
- Recall:
  - J.J. Thomson (1897): electric charge is corpuscular, "stored" in electrons.
- R. Milikan (1910): electric charge is quantized, always showing up in integral multiples of e.
- **Milikan's experiment:** measuring the charge on ionized oil droplets.

R.A. Millikan
Nobelprize.org

The oil drop experiment

- The experiment entailed balancing the downward gravitational force with the upward buoyant and electric forces on tiny charged droplets of oil suspended between two metal electrodes.
- Since the density of the oil was known, the droplets' masses could be determined from their observed radii.
- Using a known electric field, Milikan and Fletcher could determine the charge on oil droplets in mechanical equilibrium.

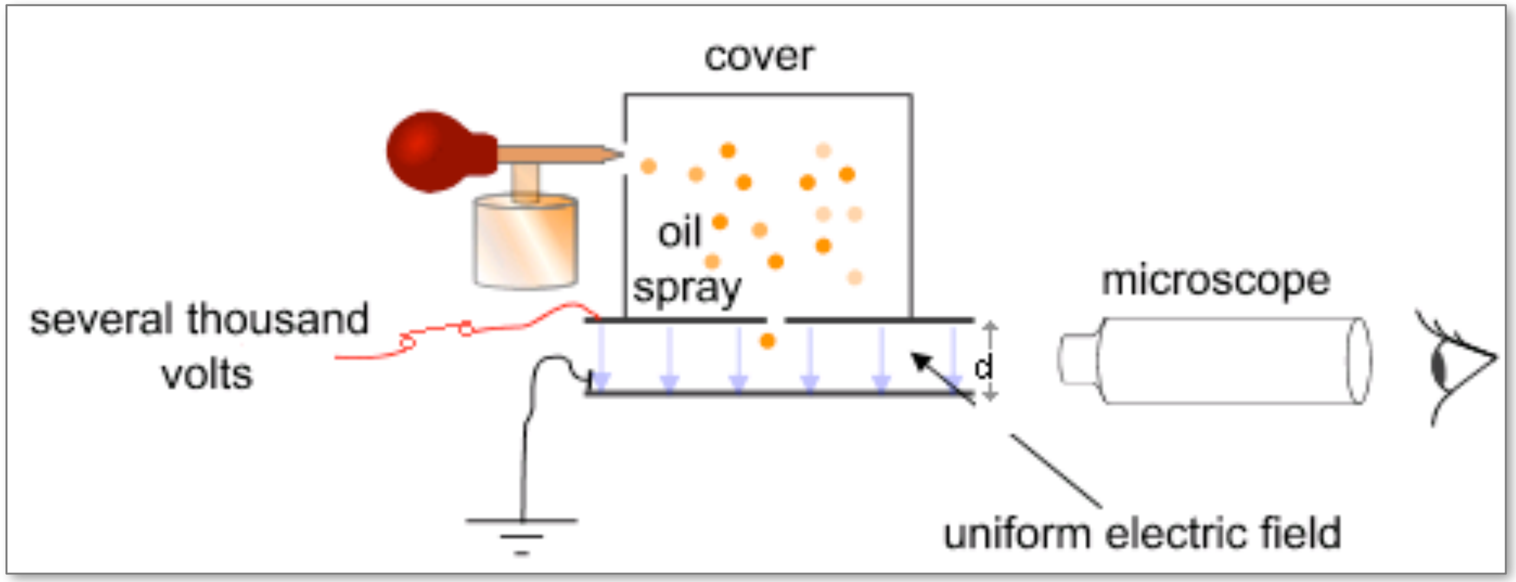


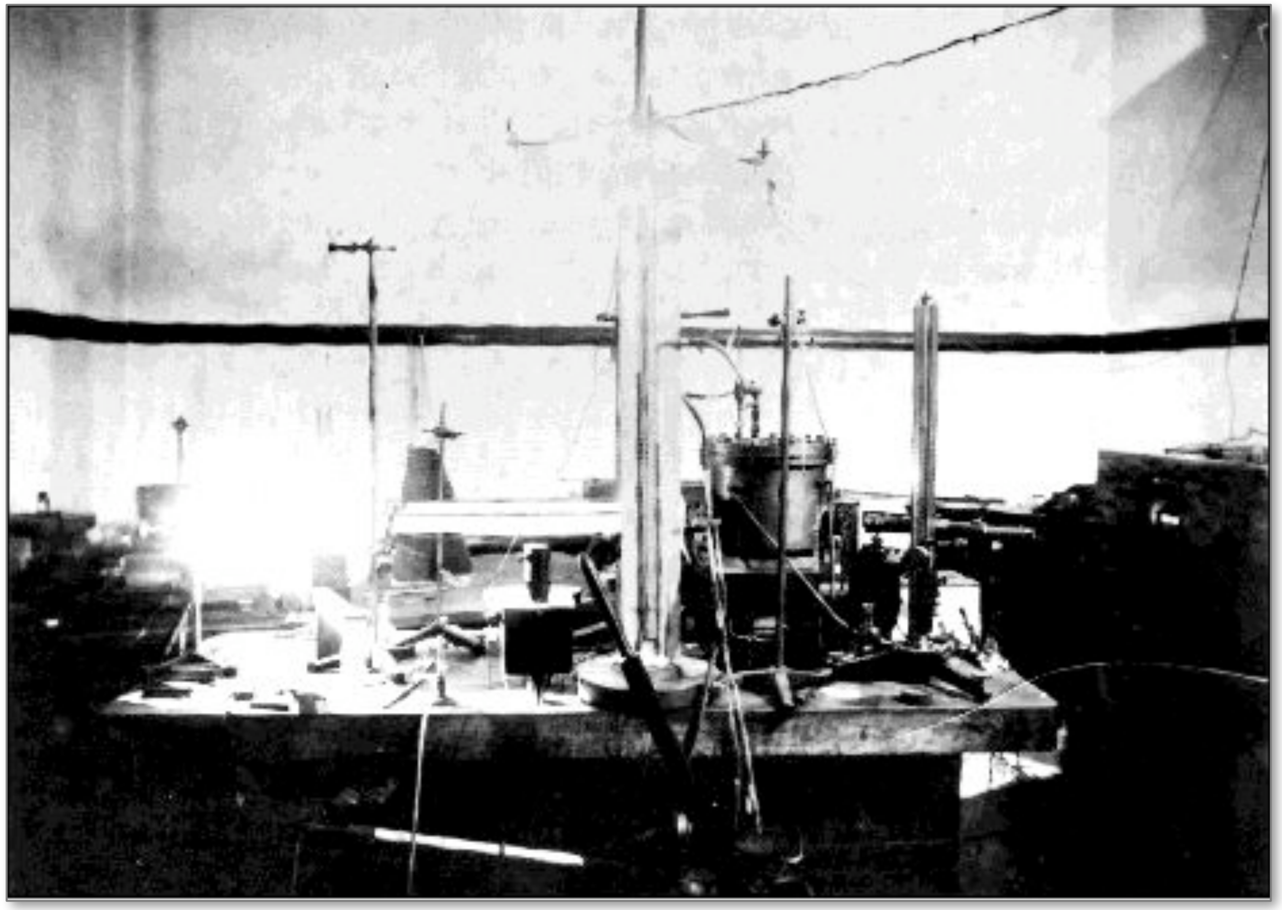


Millikan's setup

The oil drop experiment

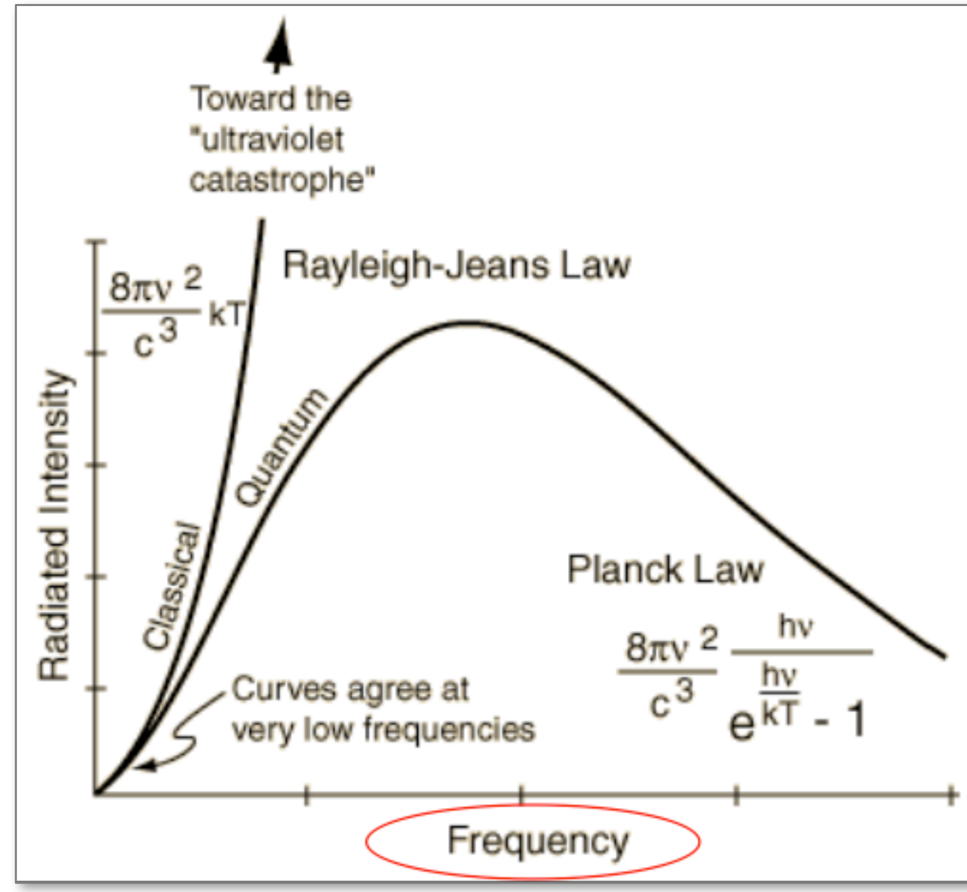
- By repeating the experiment for many droplets, they confirmed that the charges were all multiples of some fundamental value.
- They calculated it to be 1.5924(17)x10^{-19} C, within 1% of the currently accepted value of 1.602176487(40)x10^{-19} C.
- They proposed that this was the charge of a single electron.





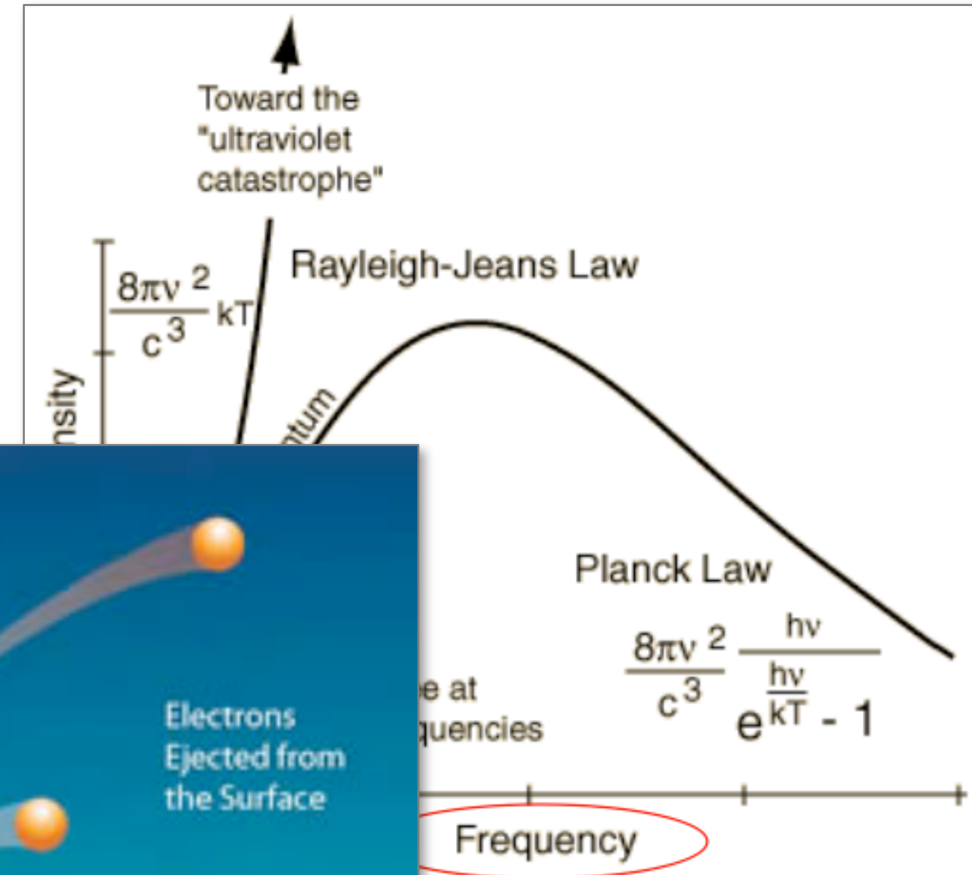
Millikan's setup

Quantization of energy



- Recall:
  - M. Planck (1900): blackbody radiation spectrum can be explained if light of frequency ν comes in quantized energy packets, with energies of hν.
- A. Einstein (1905): photoelectric effect can only be theoretically understood if light is corpuscular.
- N. Bohr (1913): discrete energy spectrum of the hydrogen atom can be explained if the electron's angular momentum about the nucleus is quantized.
  - In an atom, angular momentum mvr always comes in integral multiples of ħ = h/2π.

Quantization of energy



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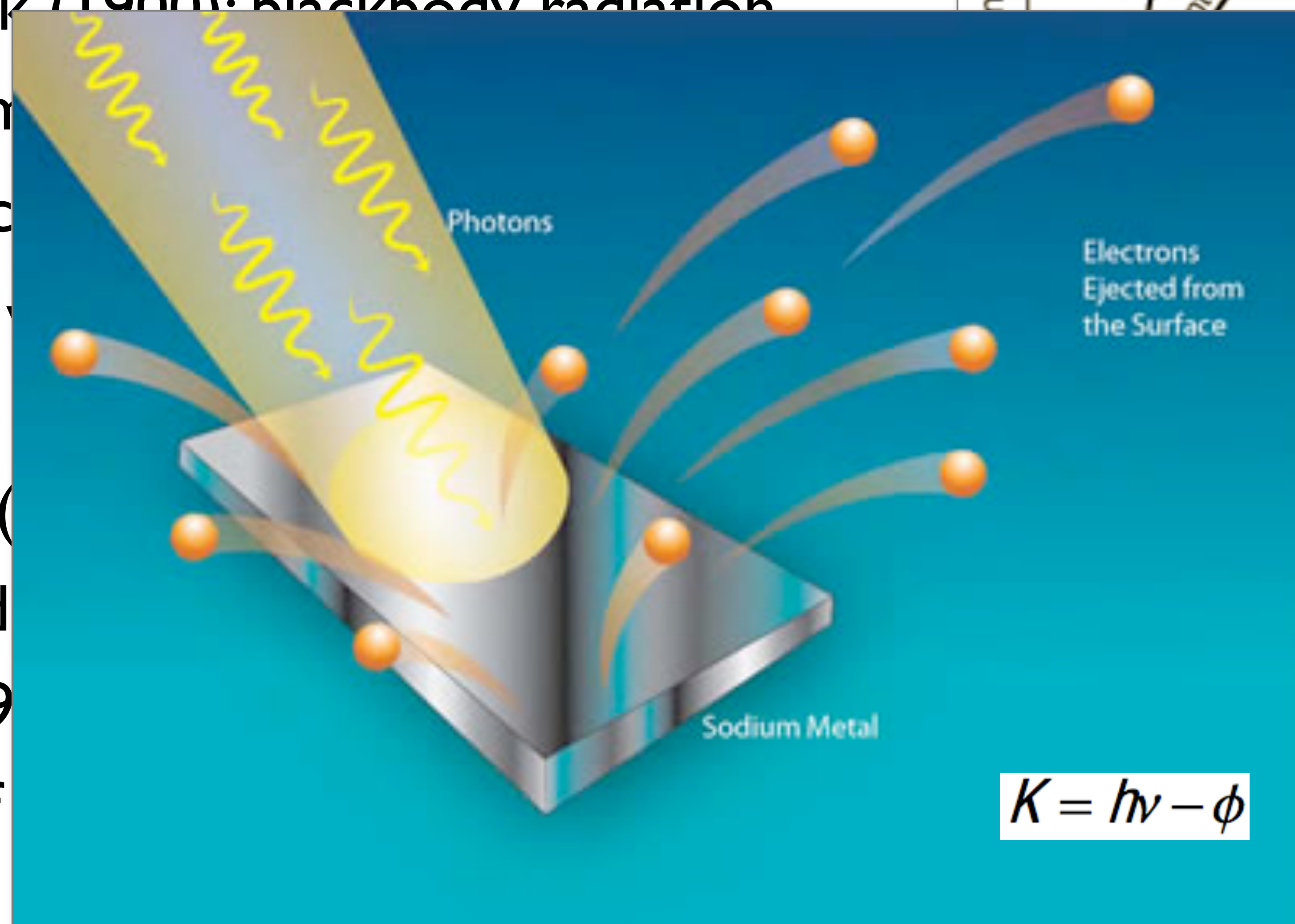

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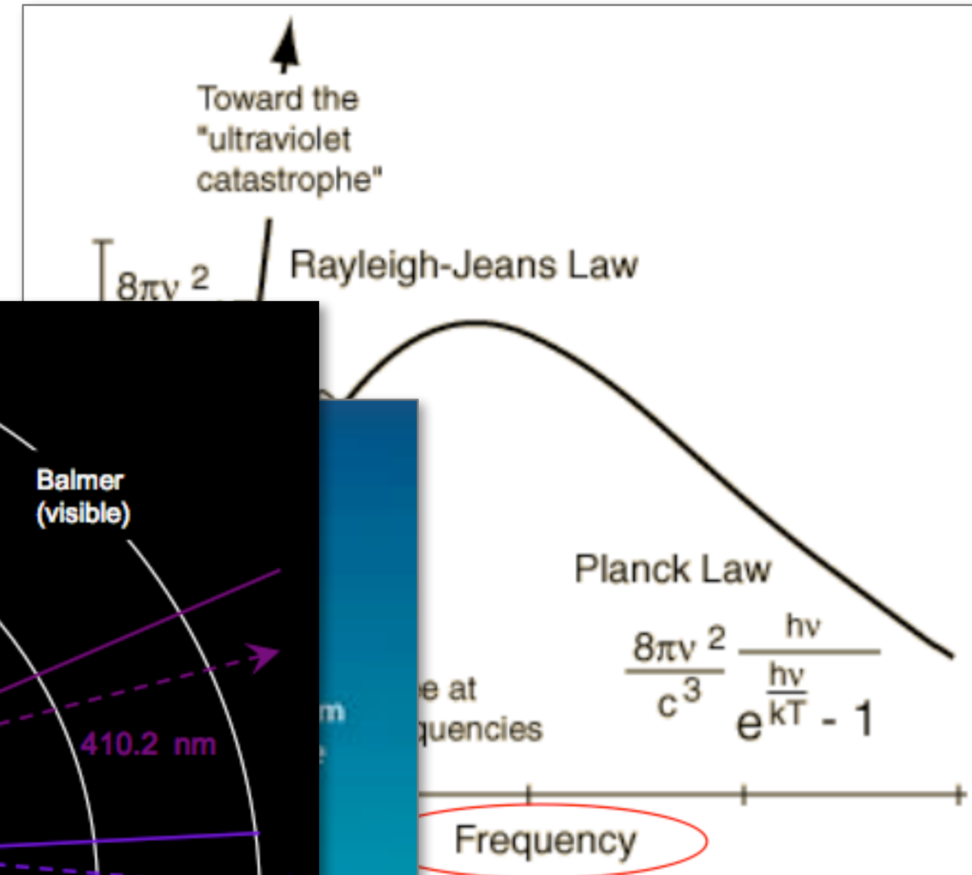
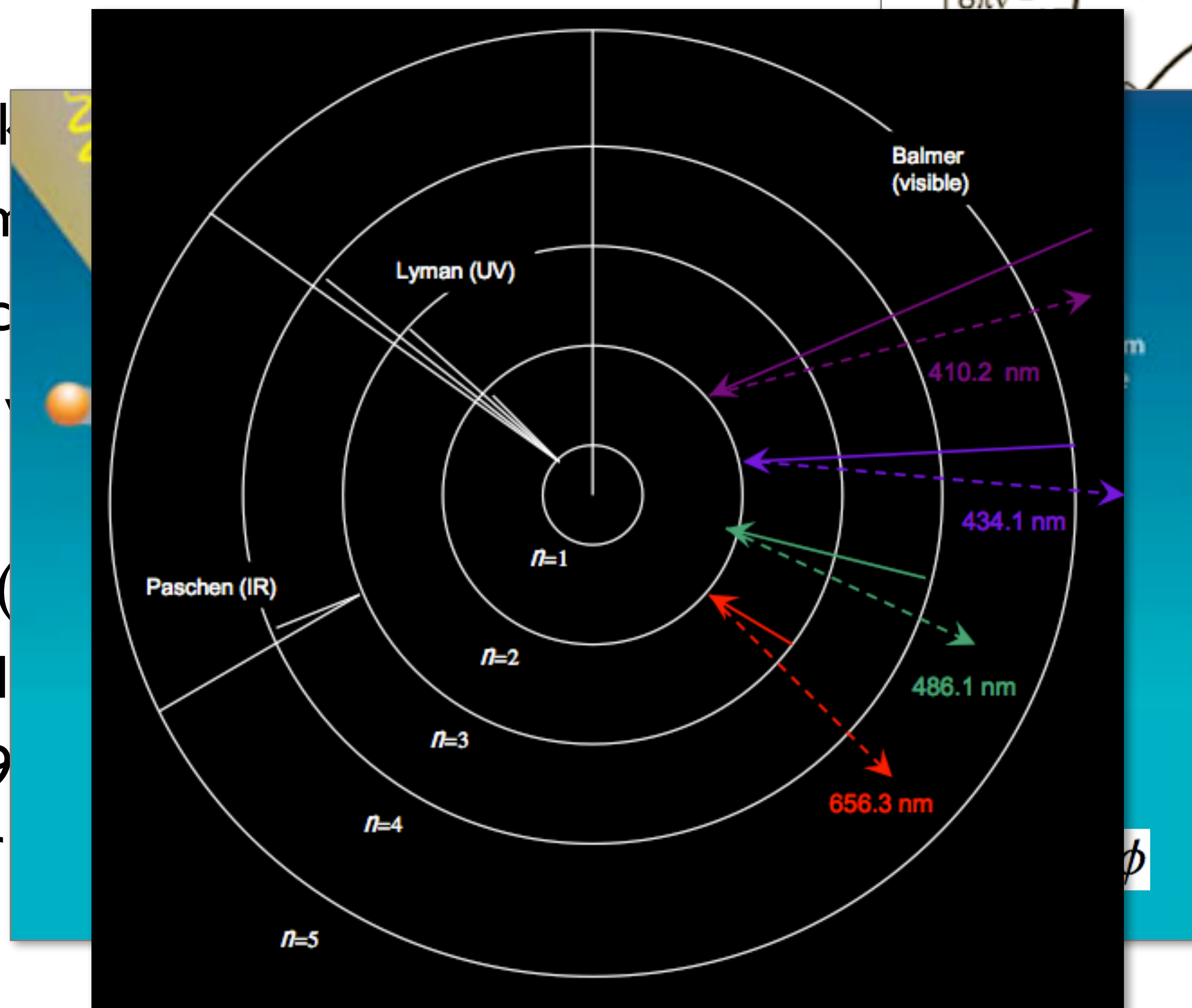
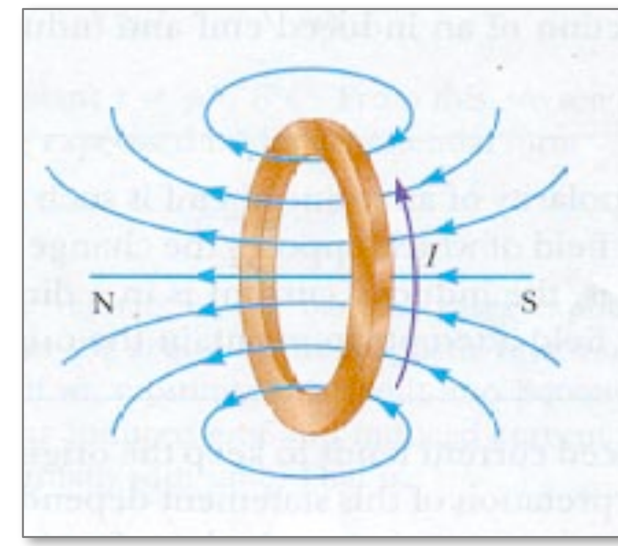




Quantization of spin

- The particle property called *spin*, is also quantized in units of ħ.
- All particles have spin: it is an inherent property, like electric charge, or mass.
- A magnetic phenomenon, spin is very important.
  - If you understand bar magnets, you (somewhat) understand spin.
- Spin is closely related to Pauli's Exclusion Principle (Spin Statistic Theorem), which relates the change in sign of the QM *wavefunction* when two identical particles are exchanged in a system (more on this later…).

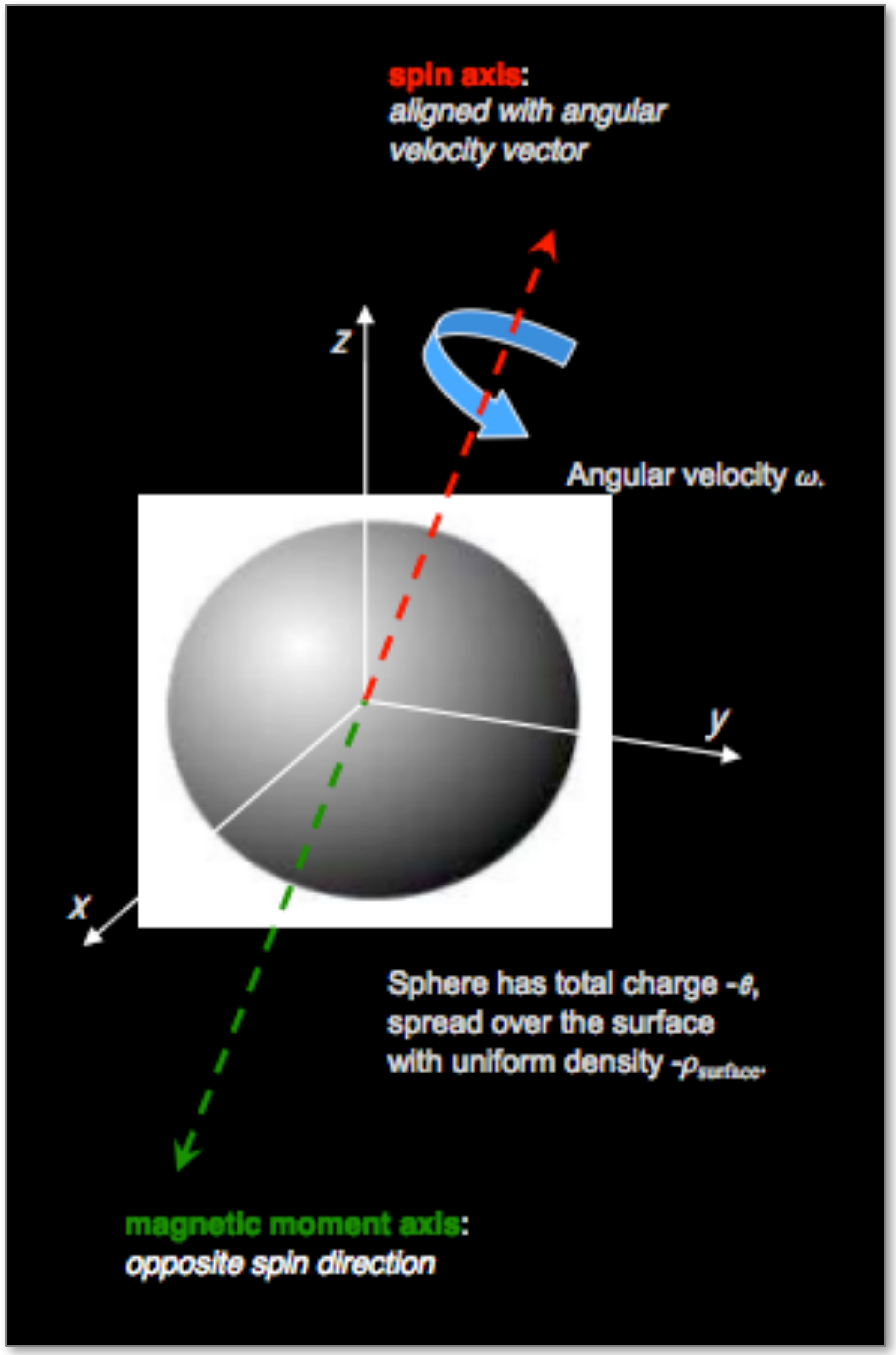


How does spin enter the picture?

- How do particles get a magnetic moment?
- Enter spin. Imagine an electron as a rotating ball of radius R, with its charge distributed over the volume of the sphere.
- The spinning sets up a current loop around the rotation axis, creating a small magnetic dipole: like a bar magnet!
- The spinning ball gets a dipole moment:

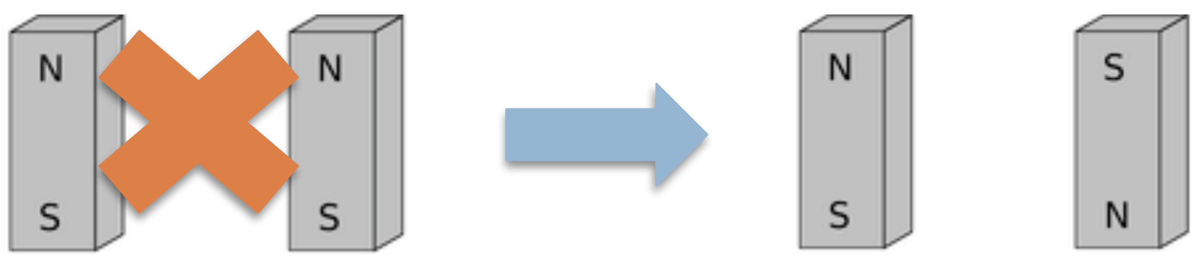
$$\vec{\mu} = -\frac{4}{3}\pi R^4 \rho \vec{\omega} \propto -\vec{S}$$

- Spin S is a vector that points along the axis of rotation (by the "right-hand rule"). The moment μ points in the opposite direction.

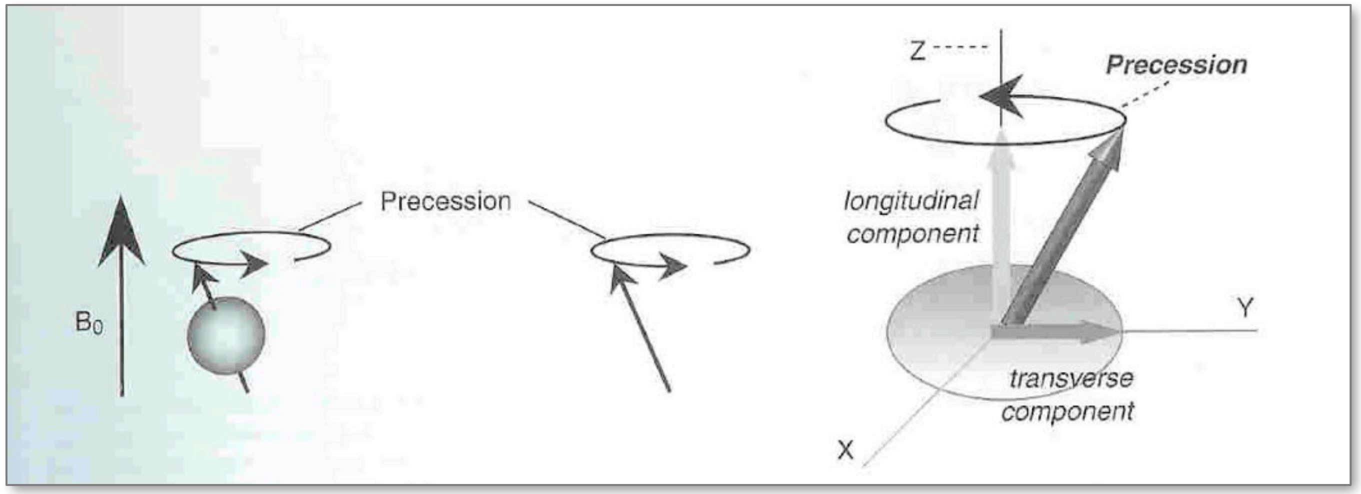


Spin-magnetism analogy

- To some extent, all elementary particles behave like tiny bar magnets, as if they had little N and S magnetic poles.



Two bar magnets set side by side will try to anti-align such that the north and south poles "match up."



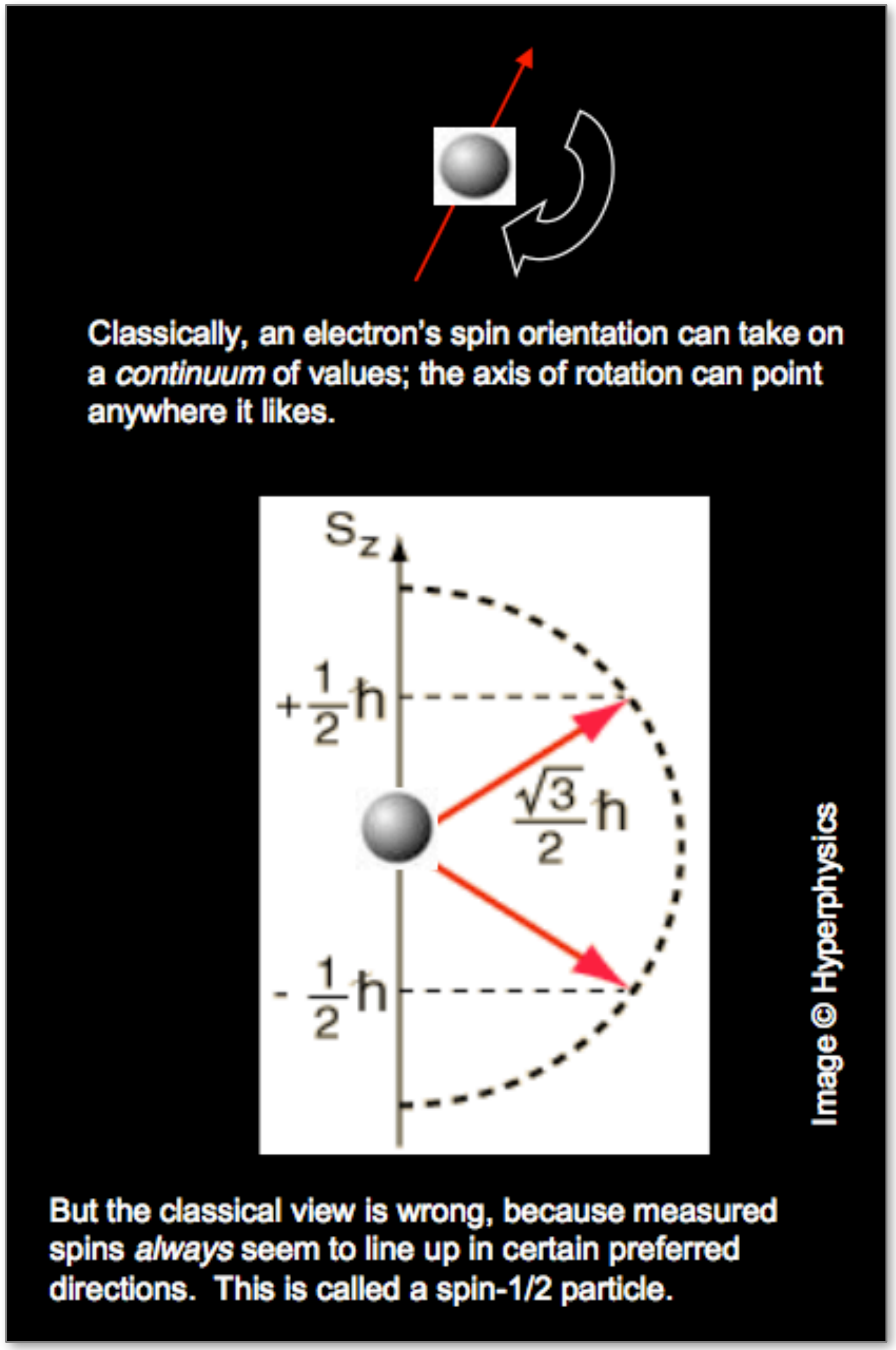
Similarly, an object with a magnetic moment will try to anti-align itself with a magnetic field.

- Jargon: if a particle behaves in this way in a magnetic field, it is said to have a non-zero magnetic dipole moment μ. In a B field, such a particle will feel a force:

$$\vec{F} = \nabla(\vec{\mu} \cdot \vec{B})$$

Quantization of spin

- If an electron is a spinning ball of charge, we understand how its magnetic moment arises.
- This is still classical physics: the spin axis may point in any direction. But, Nature is very different!
- O. Stern and W. Gerlach (1921):
  - While measuring Ag atom spins in a B field, they found that spin always aligns in two opposite directions, “up" and “down”, relative to the field.
  - Moreover, the magnitude of the spin vector is quantized in units of ħ.
- All elementary particles behave this way: their spins are always quantized, and when measured only point in certain directions (“space quantization”).



Quantization of spin

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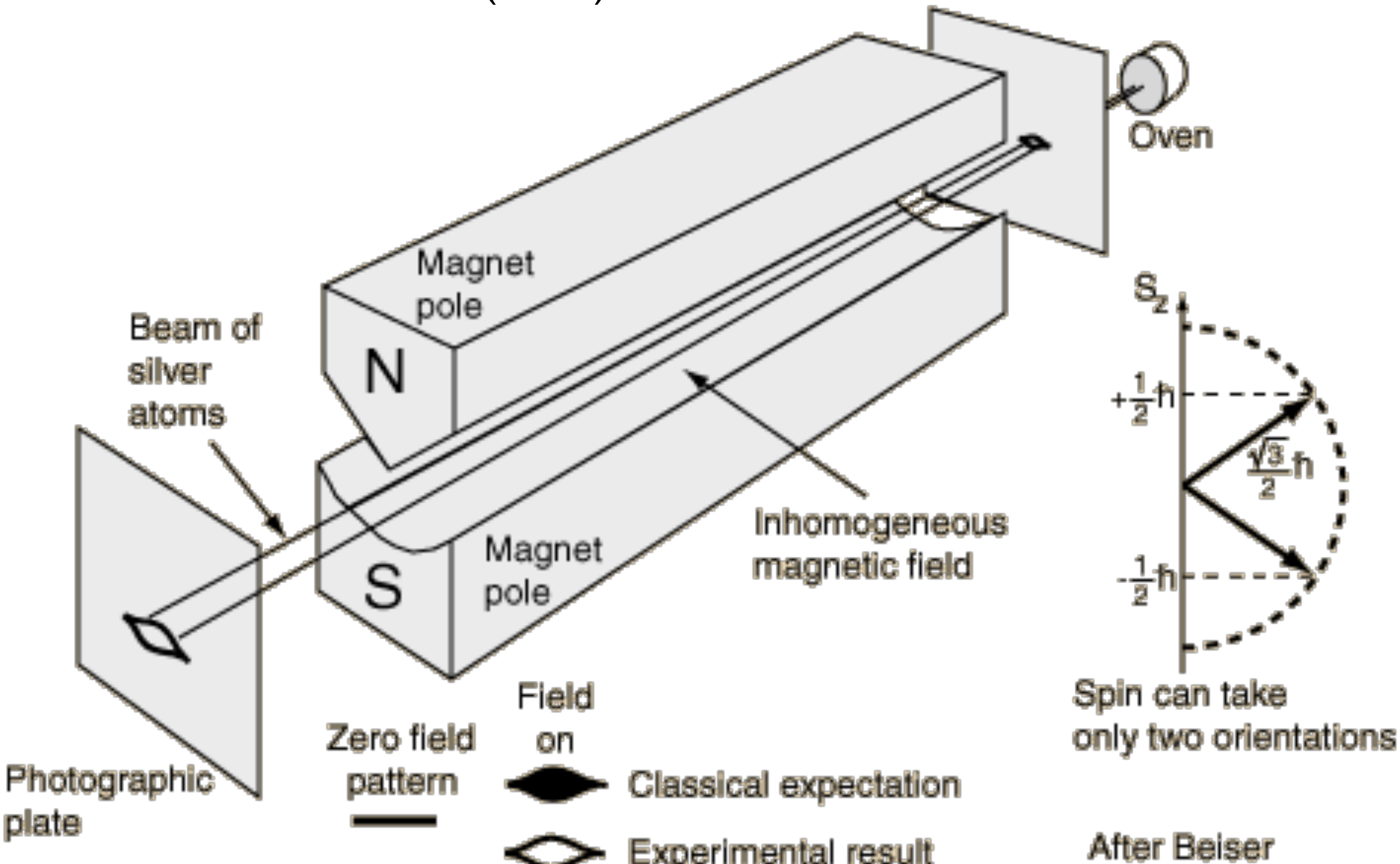

Understanding spin

- If elementary particles like the electron are actually little spinning spheres of charge, why should their spins be quantized in magnitude and direction?
- Classically, there is no way to explain this behavior.
- In 1925, S. Goudsmidt and G. Uhlenbeck realized that the classical model just cannot apply: electrons do not spin like tops; magnetic behavior must be explained some other way.
- Modern view: spin is an intrinsic property of all elementary particles, like charge or mass.
  - It is a completely quantum phenomenon, with no classical analog.

Like most other quantum mechanical properties, allowed spin values are restricted to certain numbers proportional to $\hbar$.
The classically expected continuum of values is not observed.

Quantization summarized

- General rule in QM:
  - Measurable quantities tend to come in integral (or half-integral) multiples of fundamental constants.
- Almost all of the time, Planck's constant is involved in the quantized result. It's truly a universal, fundamental constant of Nature.
- **Question:** why don't we observe quantization at macroscopic scales?

Quantization summarized

- General rule in QM:
  - Measurable quantities tend to come in integral (or half-integral) multiples of fundamental constants.
- Almost all of the time, Planck's constant is involved in the quantized result. It's truly a universal, fundamental constant of Nature.
- **Question:** why don't we observe quantization at macroscopic scales?
- **Answer:** due to the smallness of Planck's constant.
- This is analogous to Special Relativity, where the small size of the ratio v/c at everyday energy scales prevents us from observing the consequences of SR in the everyday world.

More quantum weirdness

- Observation tells us that physical quantities are not continuous down to the smallest scales, but tend to be discrete.
  - OK, we can live with that…
- But QM has another surprise: if you look small enough, matter - that is, "particles" - start to exhibit *wavelike* behavior.
- We have already seen hints of this idea.
- Light can behave like a wave, and it can behave like a particle, depending on the circumstances…

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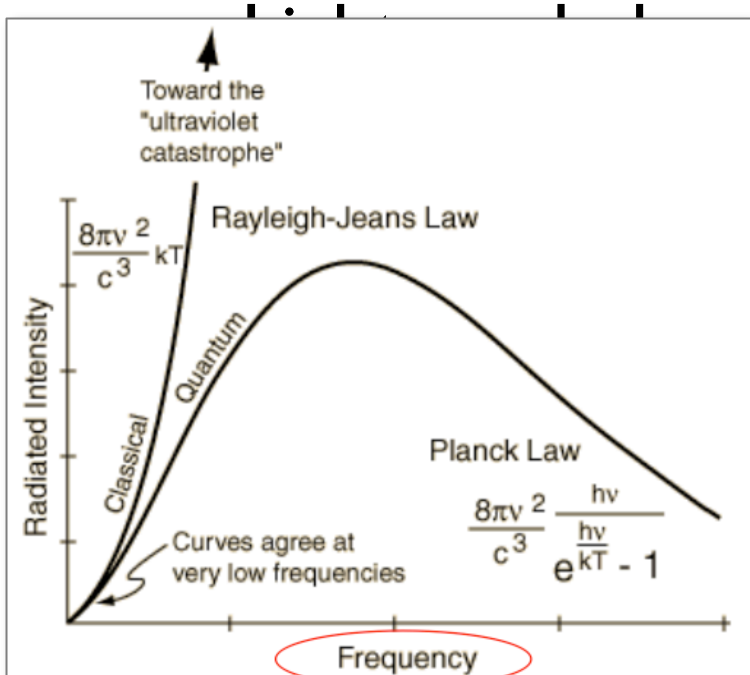
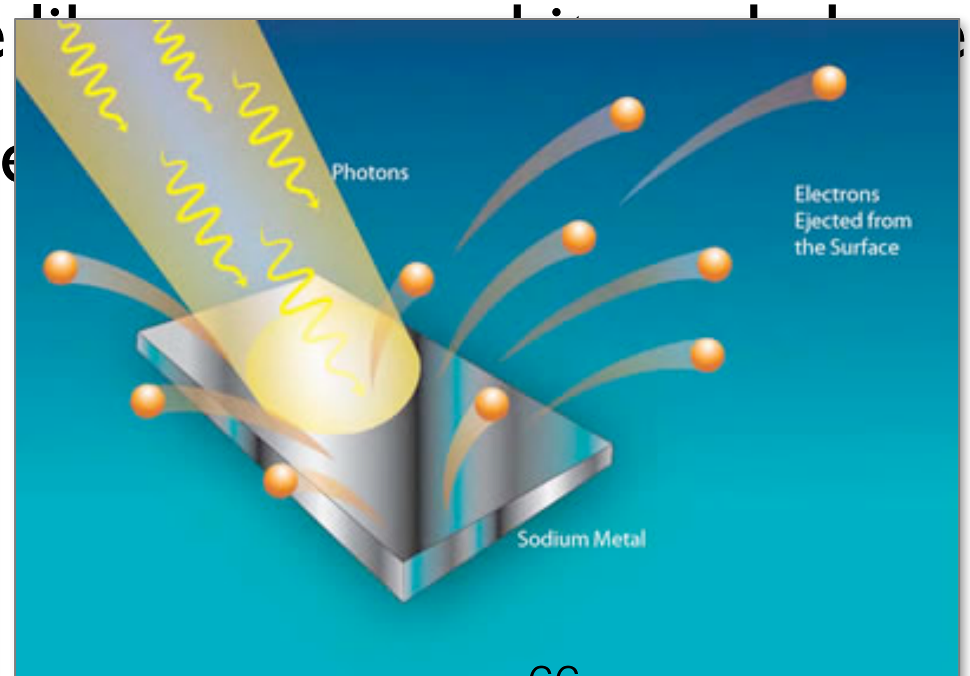
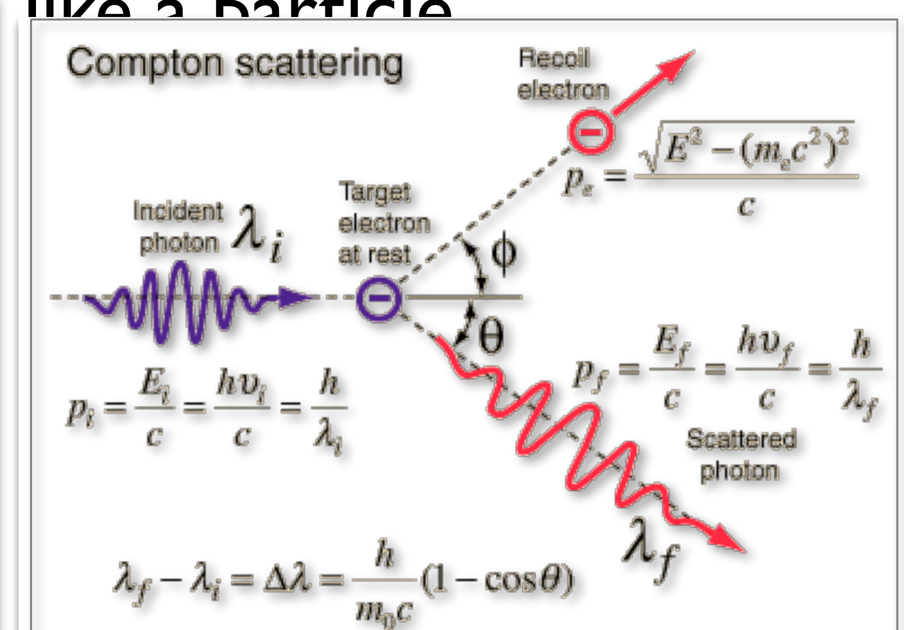






More quantum weirdness

- L. de Broglie (1924) suggested that the wave-particle behavior of light might apply equally well to matter.
- Just as a photon is associated with a light wave, so an electron could be associated with a matter wave that governs its motion.

L. de Broglie
Nobelprize.org

The de Broglie hypothesis

- de Broglie's suggestion was a very bold statement about the symmetry of Nature.
- Proposal: the wave aspects of matter are related to its particle aspects in quantitatively *the same way* that the wave and particle aspects of light are related.
- Hypothesis: for matter and radiation, the total energy E of a particle is related to the frequency ν of the wave associated with its motion by:

$$E = h\nu$$

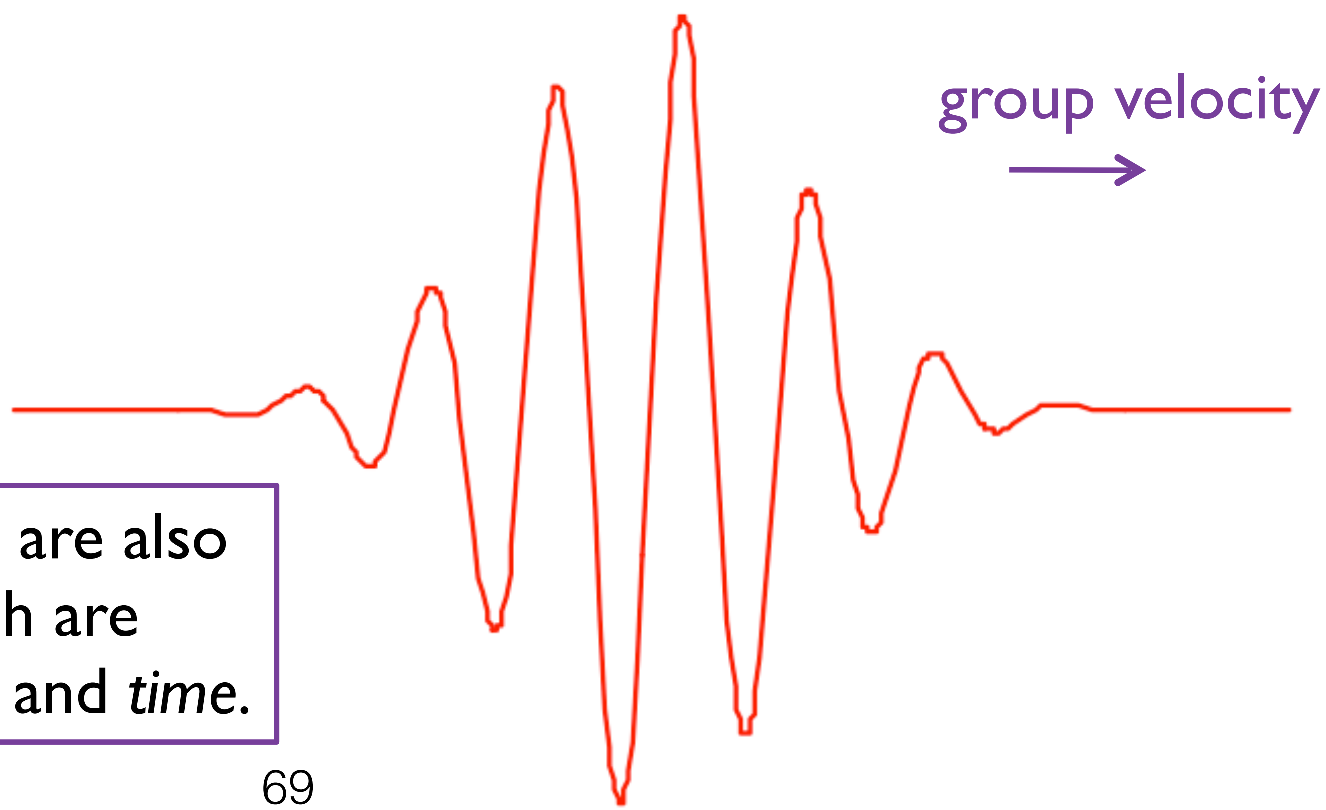
- If E=pc (recall SR), then the momentum p of the particle is related to the wavelength λ of the associated wave by the equation:

$$\mathbf{p = h/}\lambda$$

The de Broglie hypothesis

- **$p = h/\lambda$ or $\lambda = h/p$ → de Broglie relation**

- *It holds even for massive particles.*
- It predicts the de Broglie wavelength of a matter wave associated with a material particle of momentum p.



de Broglie hypothesis: particles are also associated with waves, which are extended disturbances in space and *time*.

Matter waves and the classical limit

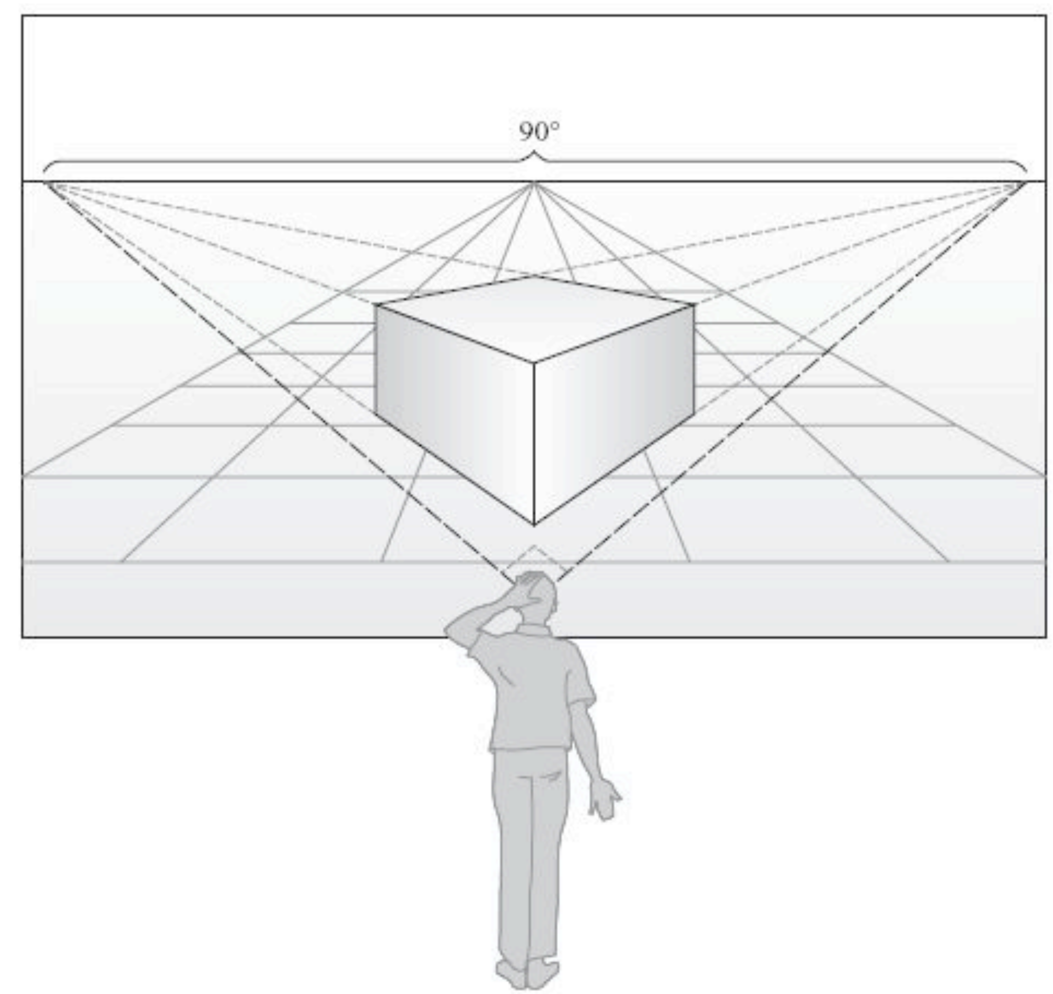
- Question: if the de Broglie hypothesis is correct, why don't macroscopic bodies exhibit wavelike behaviors?

Matter waves and the classical limit

- Question: if the de Broglie hypothesis is correct, why don't macroscopic bodies exhibit wavelike behaviors?
  - Smallness of h!

Put things in perspective?

- What is your de Broglie wavelength?
- What is the de Broglie wavelength of a 100 eV electron?

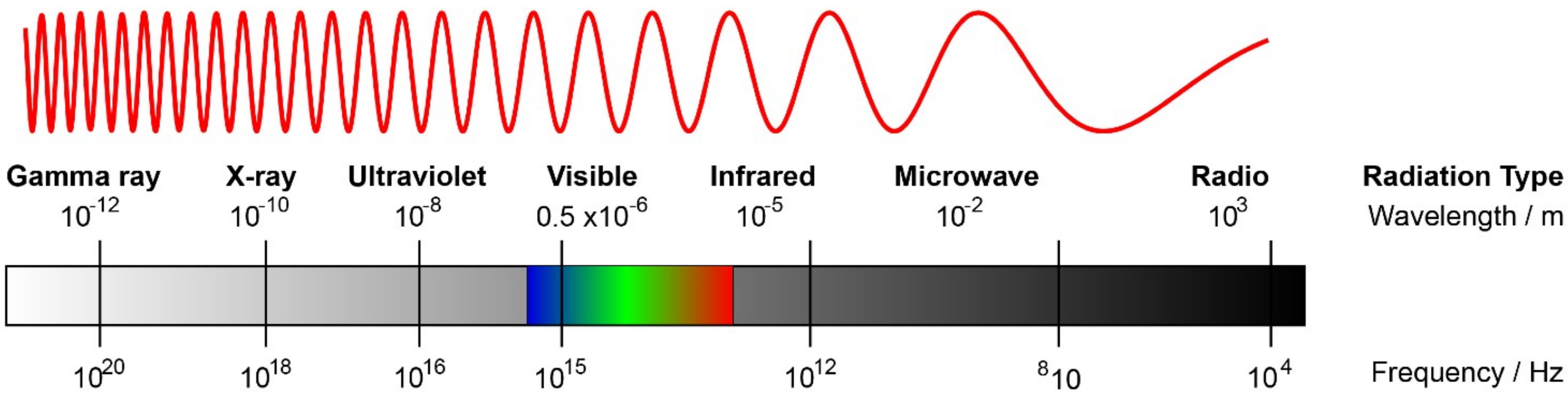


Testing the wave nature of matter

- Macroscopic particles do not have measurable de Broglie wavelengths, but electron wavelengths are about the characteristic size of X-rays.
- So, we have an easy test of the de Broglie hypothesis:
  - Check if electrons exhibit wavelike behavior (diffraction, interference, …) under the same circumstances that X-rays do.
- First, let's talk a little bit about X-rays and X-ray diffraction.

X-rays

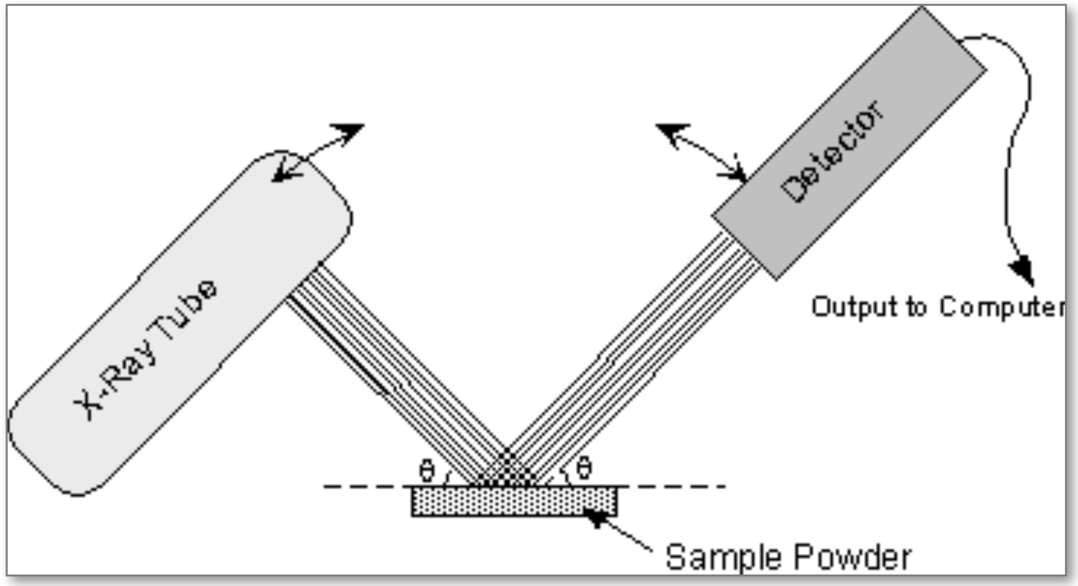
The first medical röntgengram:
a hand with buckshot, 1896.

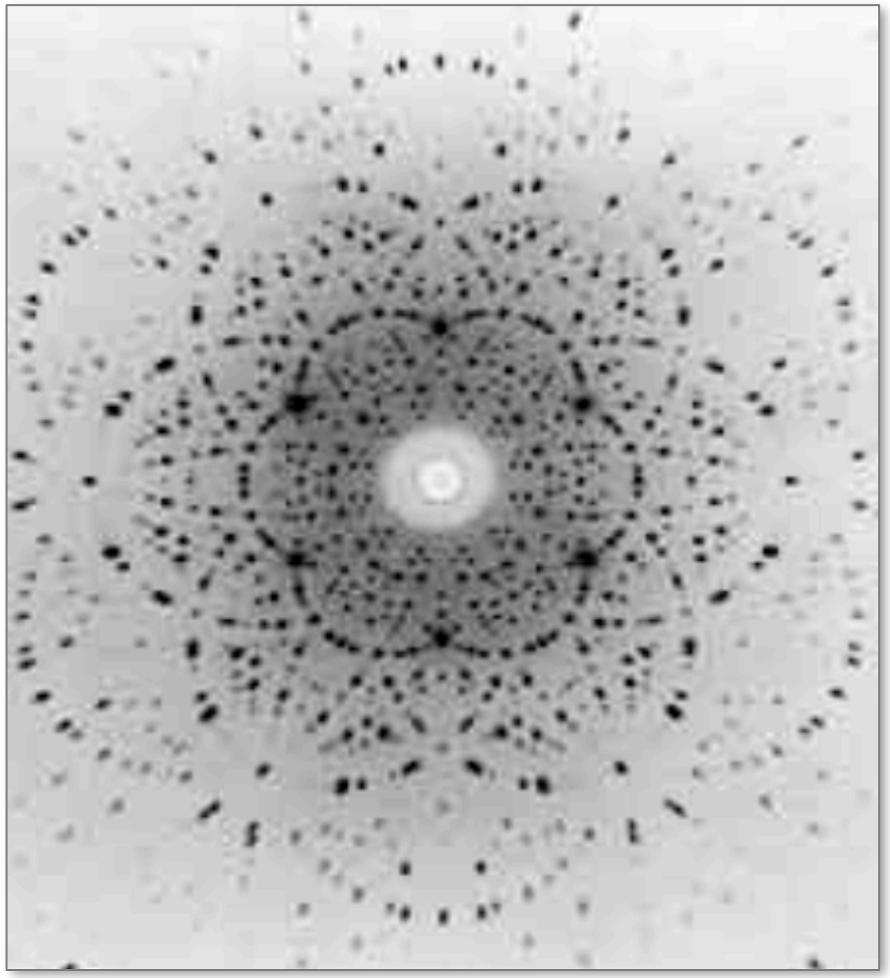


- W. Röntgen (1895): discovers X-rays, high energy photons with typical wavelengths near 0.1 nm.
- Compared to visible light (λ between 400 and 750 nm), X-rays have extremely short wavelengths and high penetrating power.
- Early in the 20th century, physicists proved that X-rays are light waves by observing X-ray diffraction from crystals.

X-ray crystallography

- How it's done: take X-rays and shoot them onto a crystal specimen.
  - Some of the X-rays will scatter backwards off the crystal.
- When film is exposed to the backscattered X-rays, geometric patterns emerge.
- In this image, the dark spots correspond to regions of high intensity (more scattered X-rays).
- The geometry of the pattern is characteristic of the structure of the crystal specimen.
  - Different crystals will create different scattering patterns.

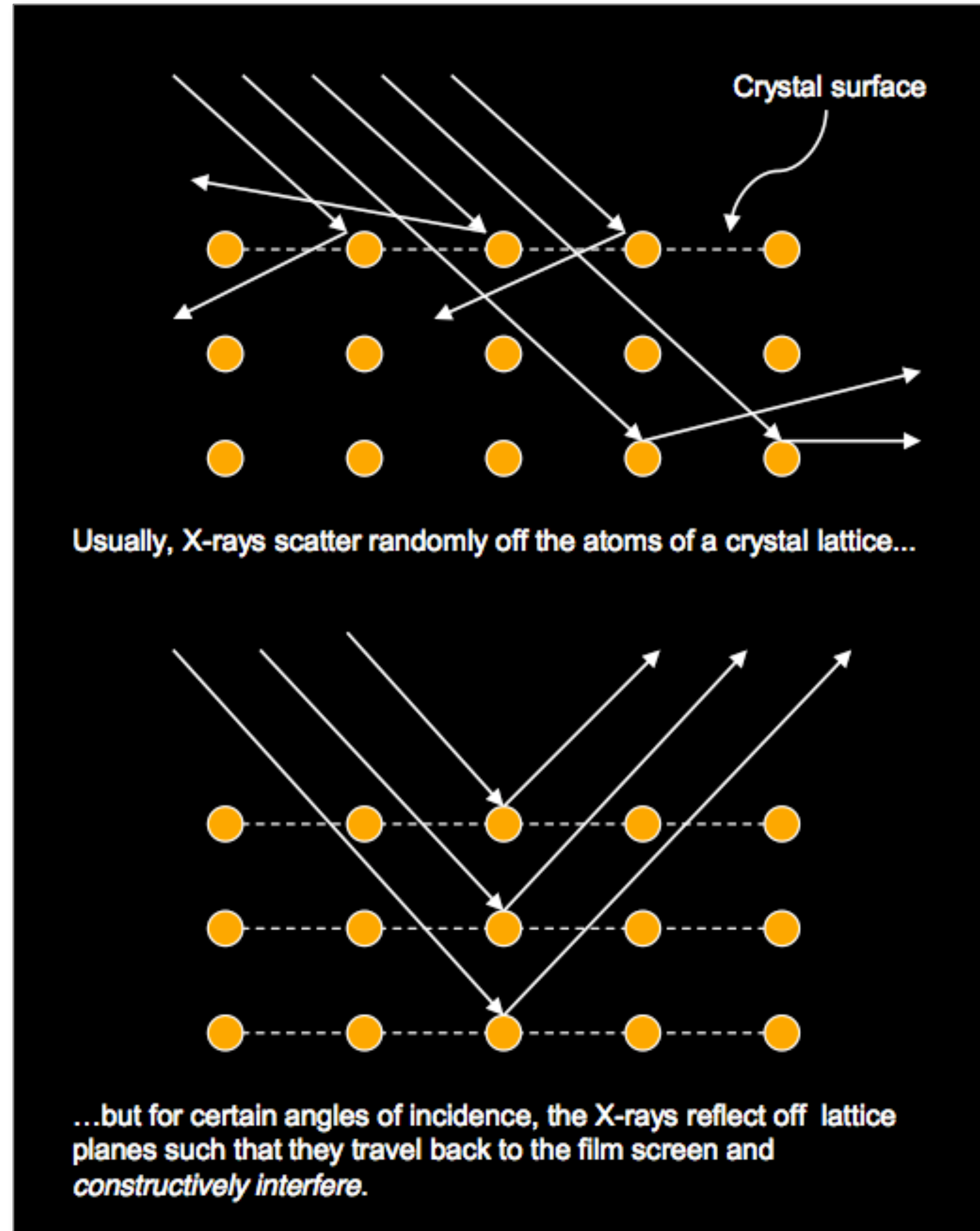




Negative image of an X-ray diffraction pattern from a beryllium aluminum silicate crystal. The X-rays seem to scatter only in preferred directions

X-ray diffraction

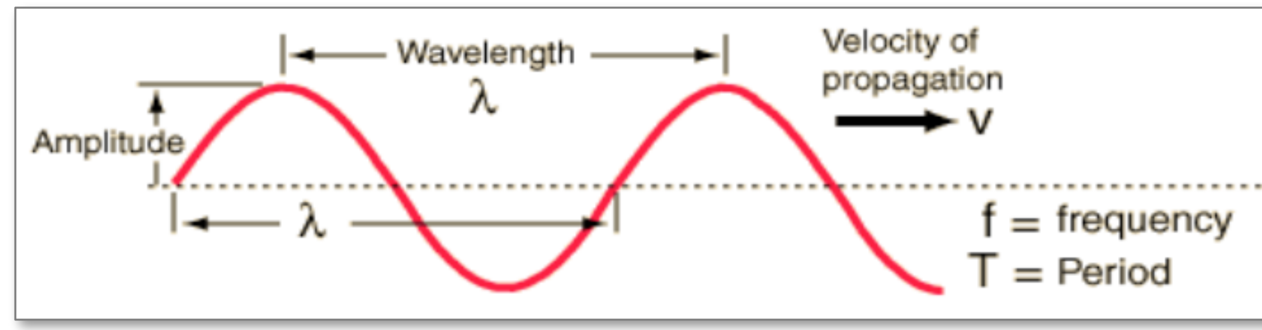
- What is the process behind X-ray crystallography?
- Why do X-rays appear to scatter off crystals in only certain directions?
  - This behavior can only be understood if X-rays are waves.
- The idea: think of the crystal as a set of semi-reflective planes.
- The X-rays reflect from different planes in the crystal, and then constructively and destructively interfere at the film screen.



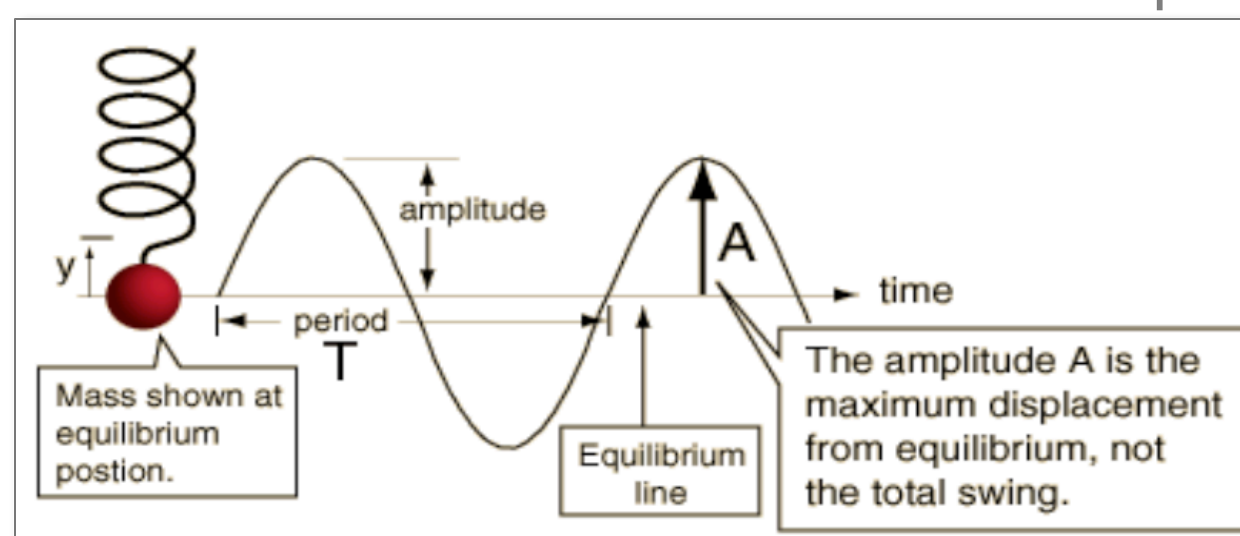
Basic wave concepts

- To understand waves in QM, let's review some basic wave concepts.
- Wavelength l: the repeat distance of a wave in space.
- Period T: the "repeat distance" of a wave in time.
- Frequency ν: the inverse of the period; ν = 1/T.
- Amplitude A: the wave's maximum displacement from equilibrium.
    - Classically, this determines a wave's intensity.

Wavelength is easy to visualize; it is the distance over which the wave starts to repeat.

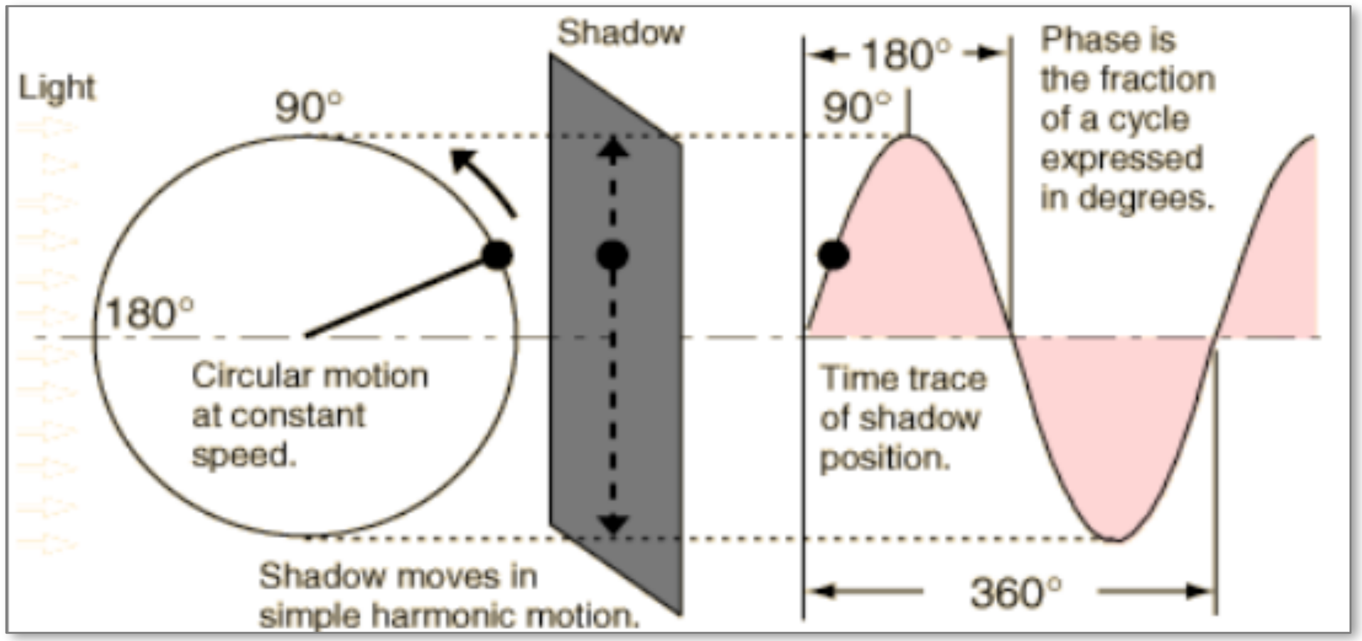


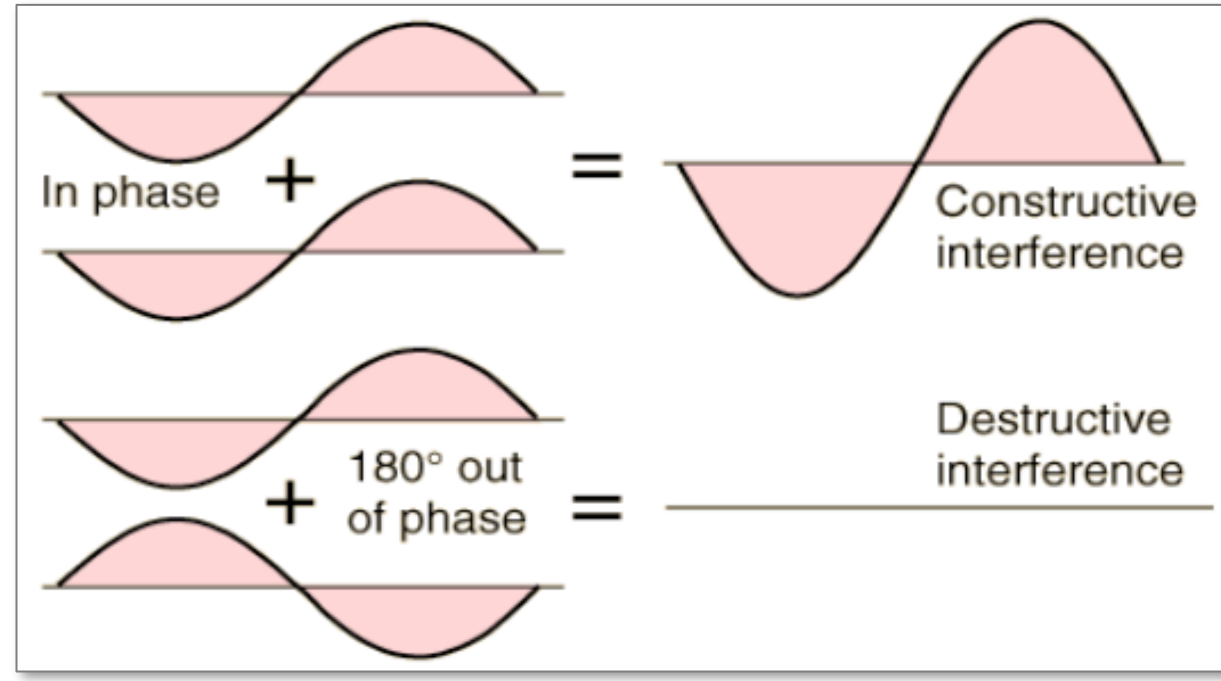
For an object executing periodic motion, like a mass on a spring, the period is just the time interval over which the wave starts to repeat.



Basic concept: interference

Waves can add or cancel, depending on their relative phase. You can visualize phase by imagining uniform circular motion on a unit circle.



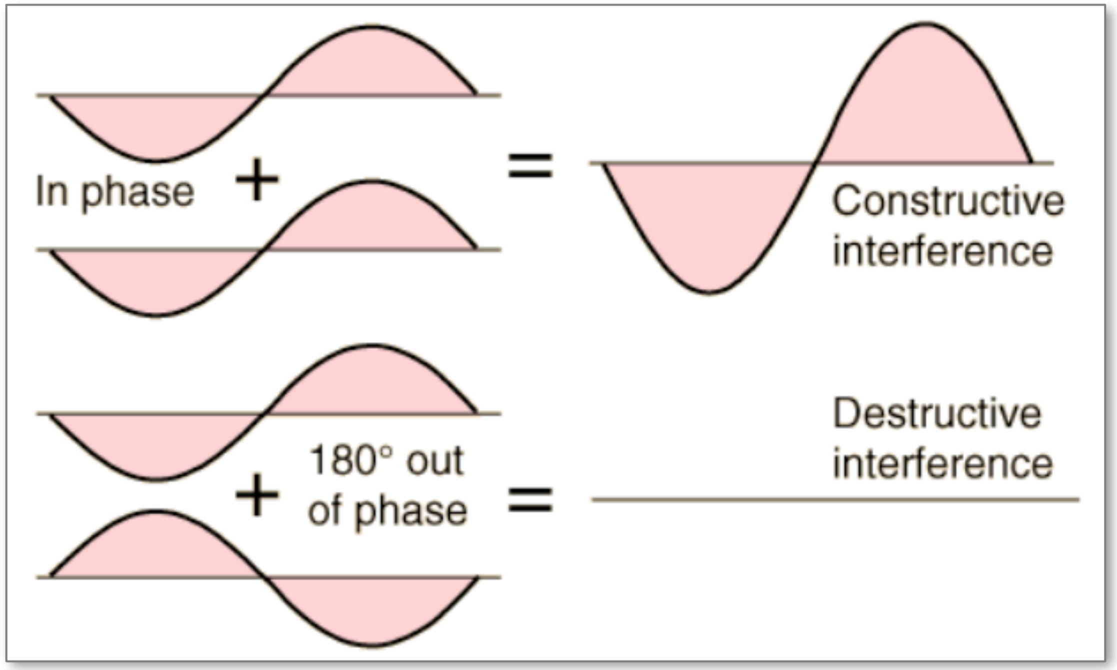


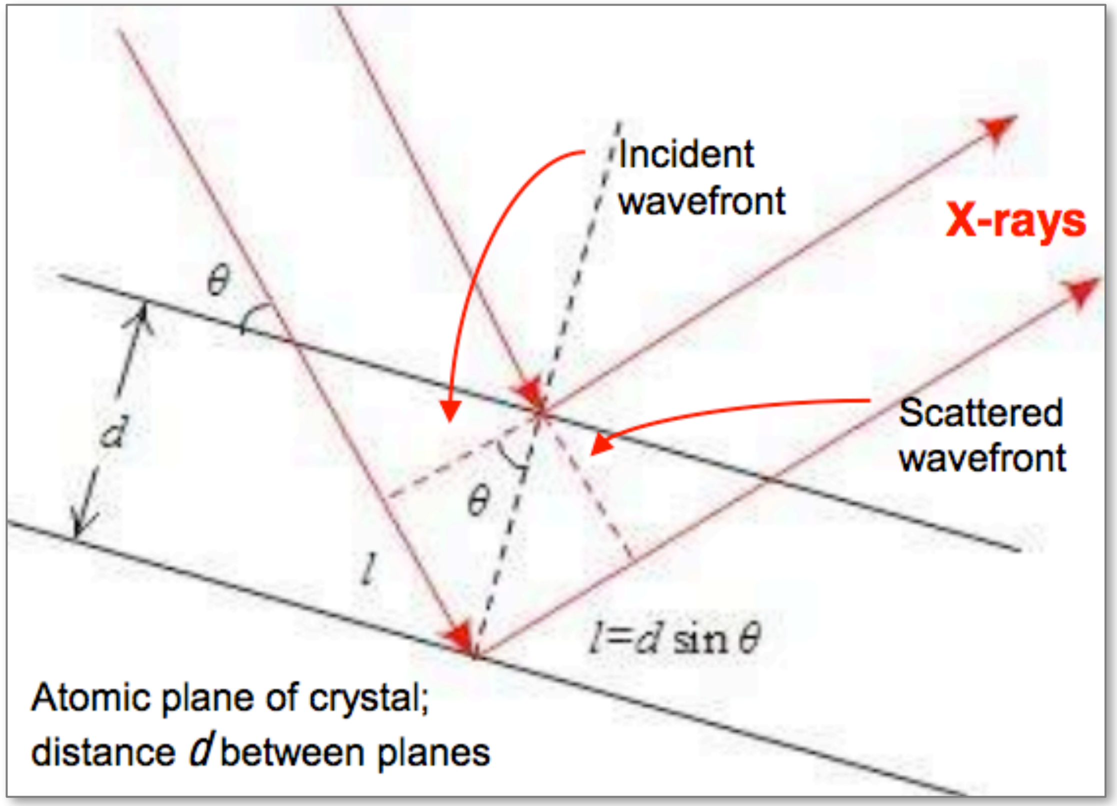
- Principle of Superposition: you can add up any number of waves (sinusoids) to get another wave.
- The resultant wave may be larger or smaller than its components, depending on their relative phase angles.
- Phases can be understood in terms of motion on the unit circle.
  - Hence, waves interfere, canceling each other at certain locations.
- Interference is what gives rise to the light and dark spots in the X-ray diffraction pattern.

How X-ray interference works

- Consider two X-ray beams reflecting from two crystal planes a distance d >> 0.1nm apart.
- The wave reflecting from the lower plane travels a distance 2l farther than the upper wave.
- If an integral number of wavelengths $n\lambda$ just fits into the distance 2l, the two beams will be in phase and will constructively interfere.
- Only certain incident angles lead to constructive interference:

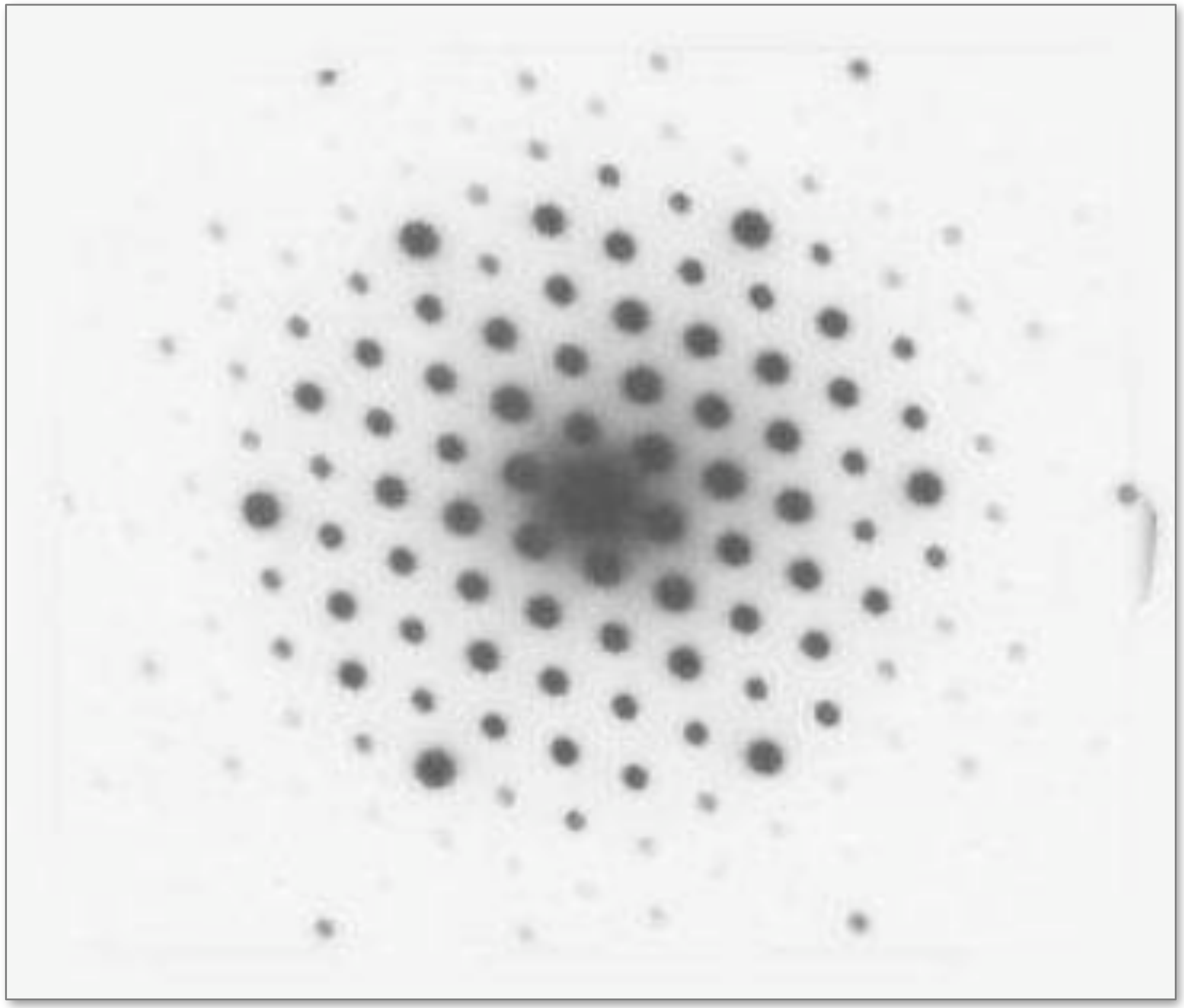
$$n\lambda = 2l = 2d\sin\theta$$





Electron diffraction

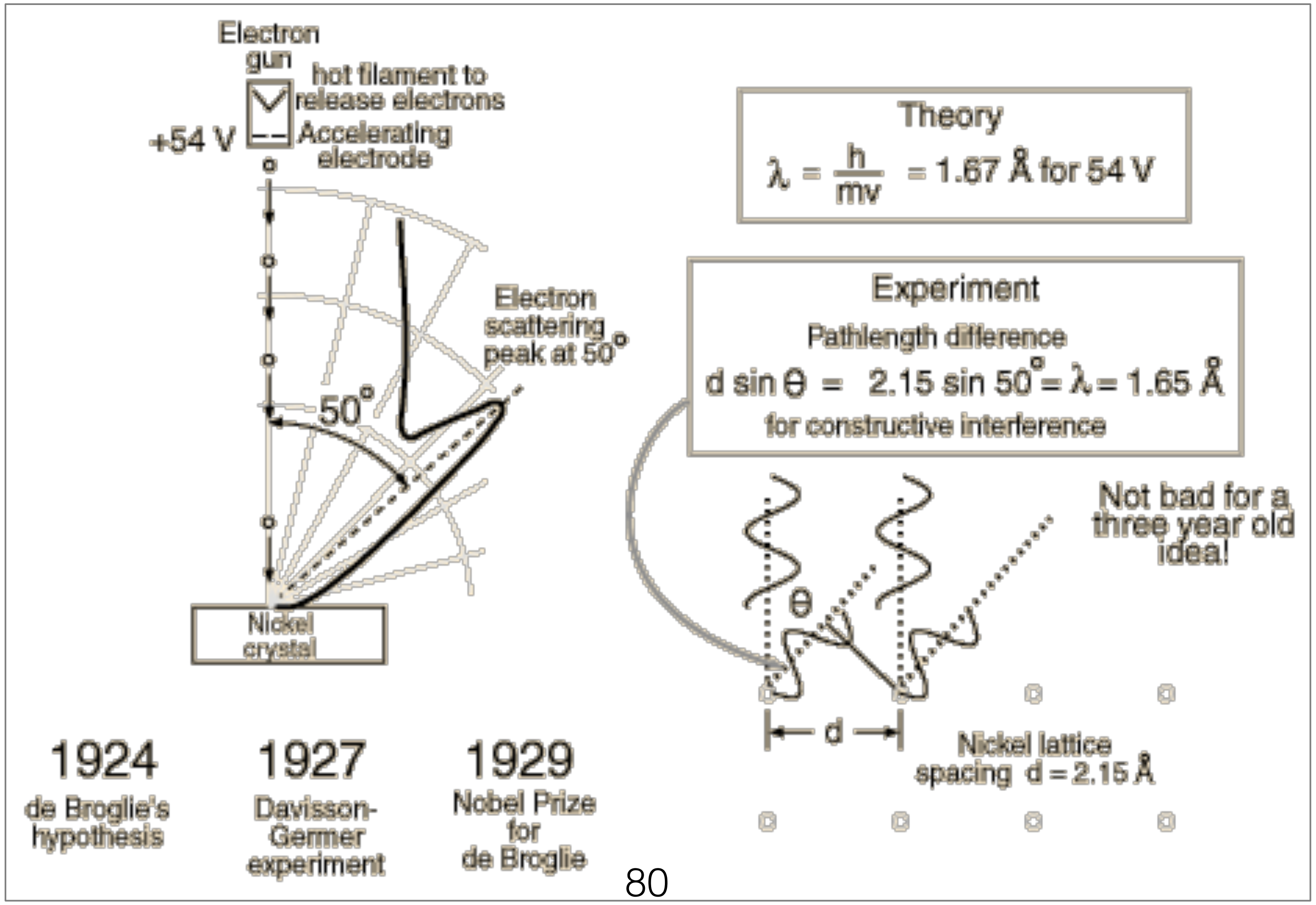
Diffraction pattern created by scattering electrons off a crystal. (This is a negative image, so the dark spots are actually regions of constructive interference.) Electron diffraction is only possible if electrons are waves.



- Scattering electrons off crystals also creates a diffraction pattern!
- Electron diffraction is only possible if electrons are waves.
- Hence, electrons (matter particles) can also behave as waves.

Observation of electron diffraction

- Electron diffraction was first observed in a famous 1927 experiment by Davisson and Germer.
- They fired 54 eV electrons at a nickel target and observed diffraction peaks consistent with de Broglie's hypothesis.



Understanding matter waves

- Let's think a little more about de Broglie waves.
- In classical physics, energy is transported either by waves or by particles.
  - Particle: a definite, localized bundle of energy and momentum, like a bullet that transfers energy from gun to target.
  - Wave: a periodic disturbance spread over space and time, like water waves carrying energy on the surface of the ocean.
- In quantum mechanics, the same entity can be described by both a wave and a particle model:
  - Electrons scatter like localized particles, but they can also diffract like extended waves.

The wave function

- Okay, particles ↔ waves.
  - How do we represent wave-particles mathematically?
- Classically: a particle can be represented solely by its position x and momentum p. If it is acted on by an external force, we can find x and p at any time by solving a differential equation like:

$$m\frac{d^2x}{dt^2} = \frac{dp}{dt} = F \quad \text{(Newton's 2nd Law)}$$

- A wave is an extended object, so we can't represent it by the pair of numbers (x,p).
- Instead, for a wave moving at velocity $c=\lambda\nu$, we define a "wave function" $\psi(x,t)$ that describes the wave's extended motion in time and space.
- To find $\psi(x,t)$ at any position and time, we solve a differential equation like:

$$\frac{\partial^2\psi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} \quad \text{(wave equation)}$$

Something similar applies to particles…

Wave equation for quantum waves

E. Schrodinger
Nobelprize.org

- Time evolution of waves can usually be found by solving a wave equation like:

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2}$$

- E. Schrödinger (1926) found that quantum particles moving in a 1D potential V(x) obey the partial differential equation:

$$i\hbar\frac{\partial \psi(x,t)}{\partial t} = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right)\psi(x,t)$$

- Notice anything?

Wave equation for quantum waves

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✓ This expression involves ħ, the usual QM parameter.

✓ It also involves the number i, which means that the waves ψ(x,t) are complex.

The de Broglie wave

- Particles can be described by de Broglie waves of wavelength λ=h/p.
- For a particle moving in the x direction with momentum p=h/λ and energy E=hν, the wave function can be written as a simple sinusoid of amplitude A:

$$\psi(x,t) = A\sin 2\pi\left(\frac{x}{\lambda} - \nu t\right)$$

- The propagation velocity c of this wave is:

$$c = \lambda\nu$$

Quantum waves summarized

- An elementary particle like a photon can act like a particle (Compton effect, photoelectric effect) or a wave (diffraction), depending on the type of experiment / observation.
- If it's acting like a particle, the photon can be described by its position and momentum x and p. If it's acting like a wave, we must describe the photon with a wave function $\psi(x,t)$.
- Two waves can always superpose to form a third: $\psi = \psi_1 + \psi_2$.
  - This is what gives rise to interference effects like diffraction.
- The wave function for a moving particle (a traveling wave) has a simple sinusoidal form.
  - But how do we interpret $\psi(x,t)$ physically?

Interpretation of the wave function

- We have seen that any wave can be described by a wave function $\psi(x,t)$.
- For any wave, we define the wave's intensity I to be:

$$I = |\psi(x,t)|^2 \equiv \int dx\ \psi(x,t)^* \psi(x,t)$$

where the asterisk signifies complex conjugation. Note that for a plane wave this is the square of A (a constant):

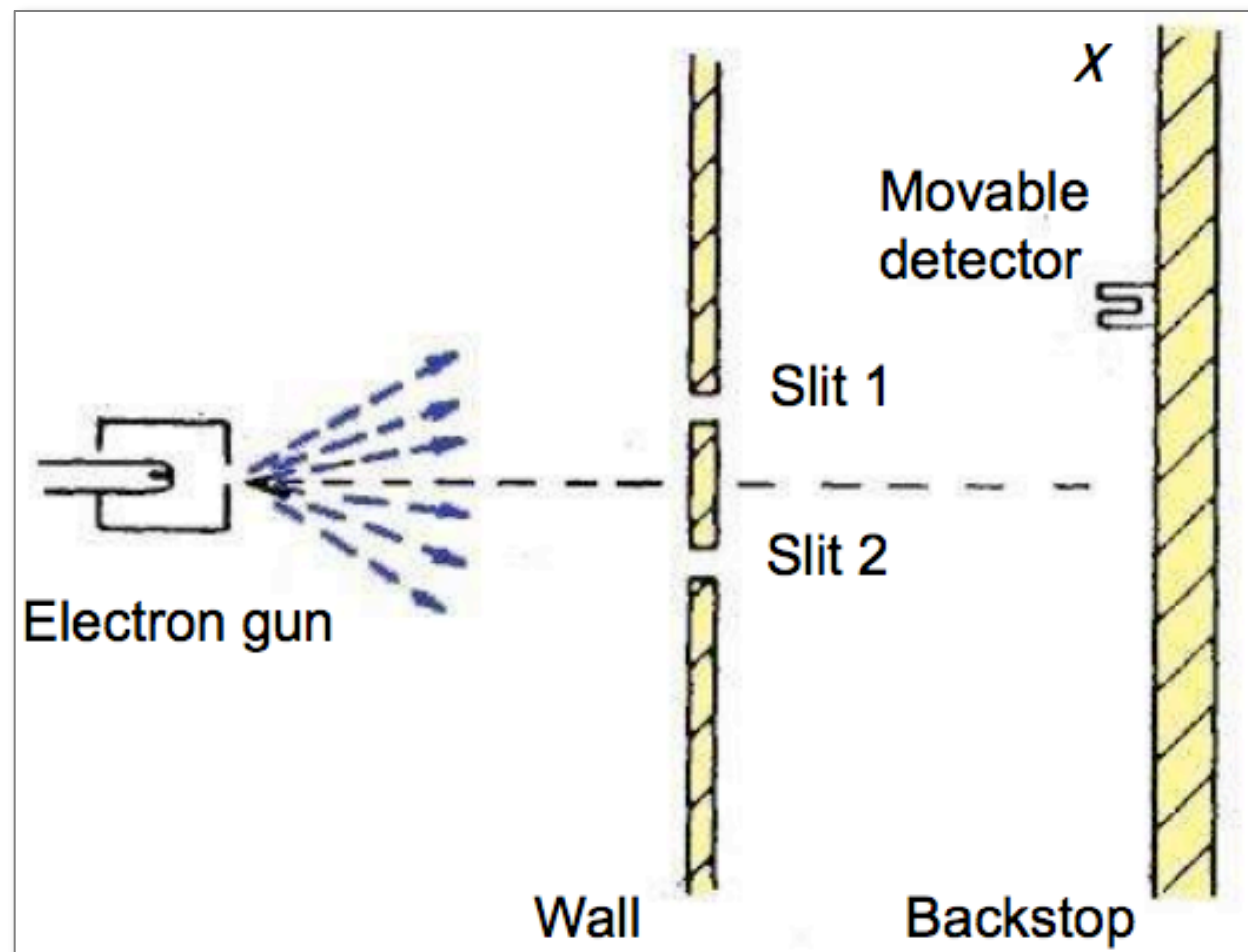
$$I = |\psi(x,t)|^2 = |A|^2$$

- Let's use the concept of intensity and a simple thought experiment to get some intuition about the physical meaning of the de Broglie wave function.

The double slit experiment

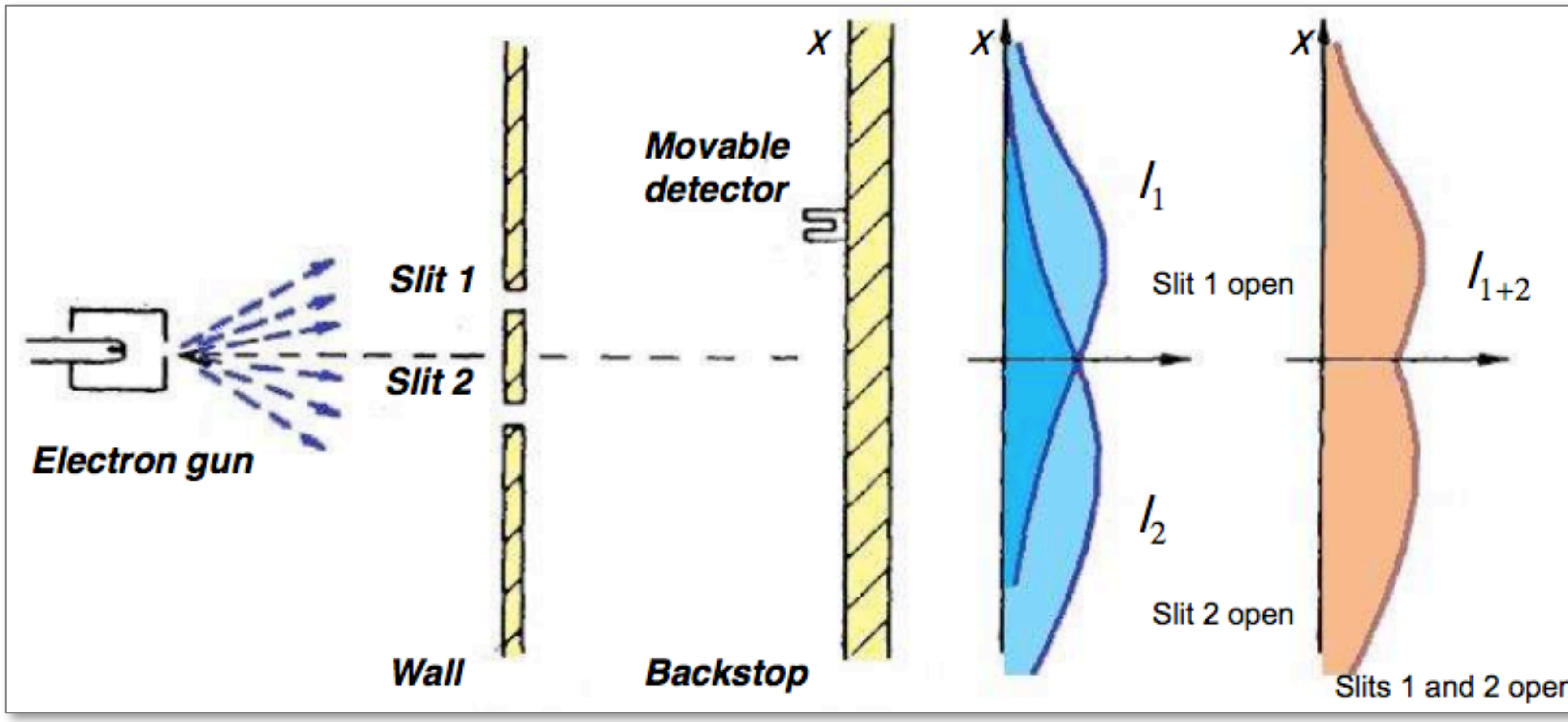
- Experiment: a device sprays an electron beam at a wall with two small holes in it. The size of the holes is close to the electrons' wavelength λ_{dB}.

- Behind the wall is a screen with an electron detector.
- As electrons reach the screen, the detector counts up how many electrons strike each point on the wall.
- Using the data, we plot the intensity I(x): the number of electrons arriving per second at position x.



The double slit experiment

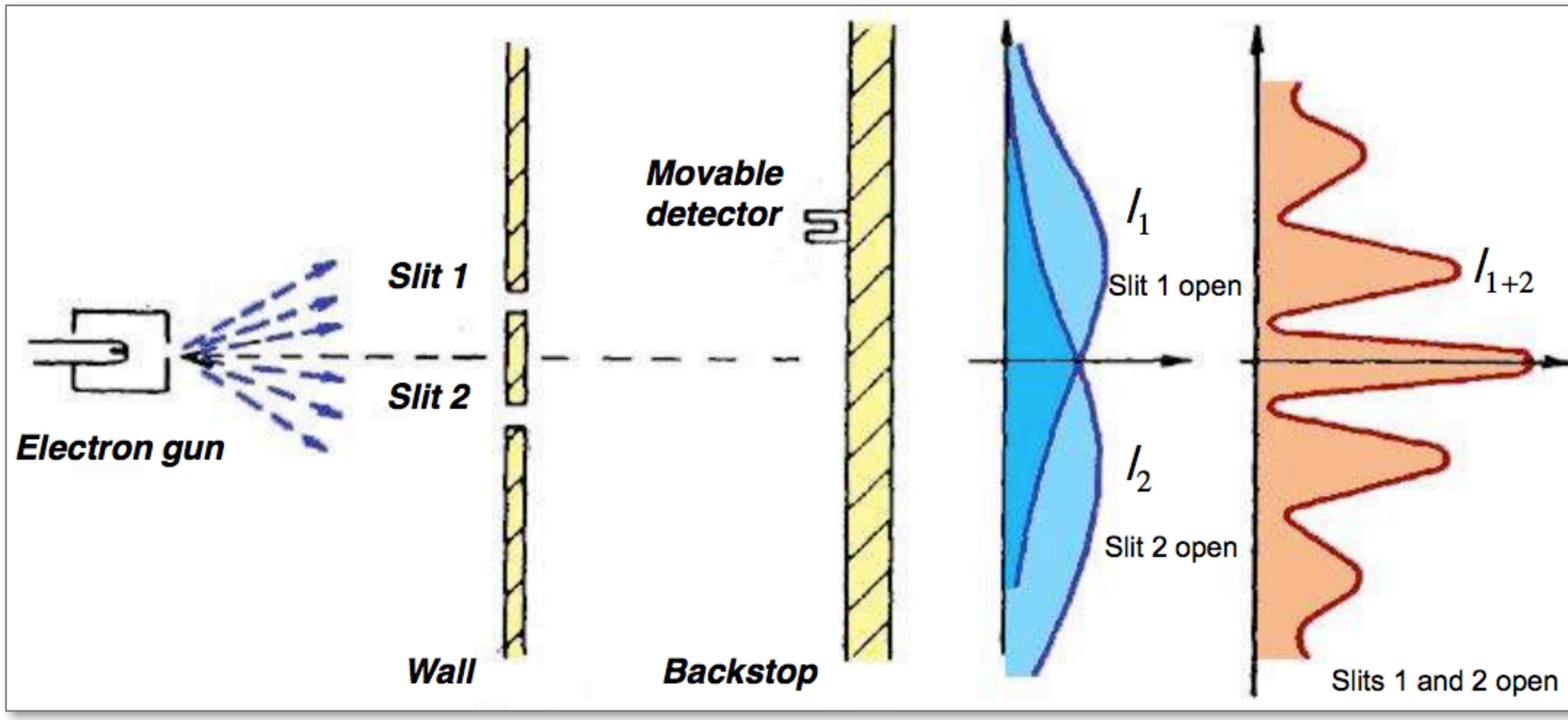
- The classical result:



- If electrons are classical particles, we expect the intensity in front of each slit to look like a bell curve, peaked directly in front of each slit opening.
- When both slits are open, the total intensity I_{1+2} on the screen should just be the sum of the intensities I_1 and I_2 when only one or the other slit is open.

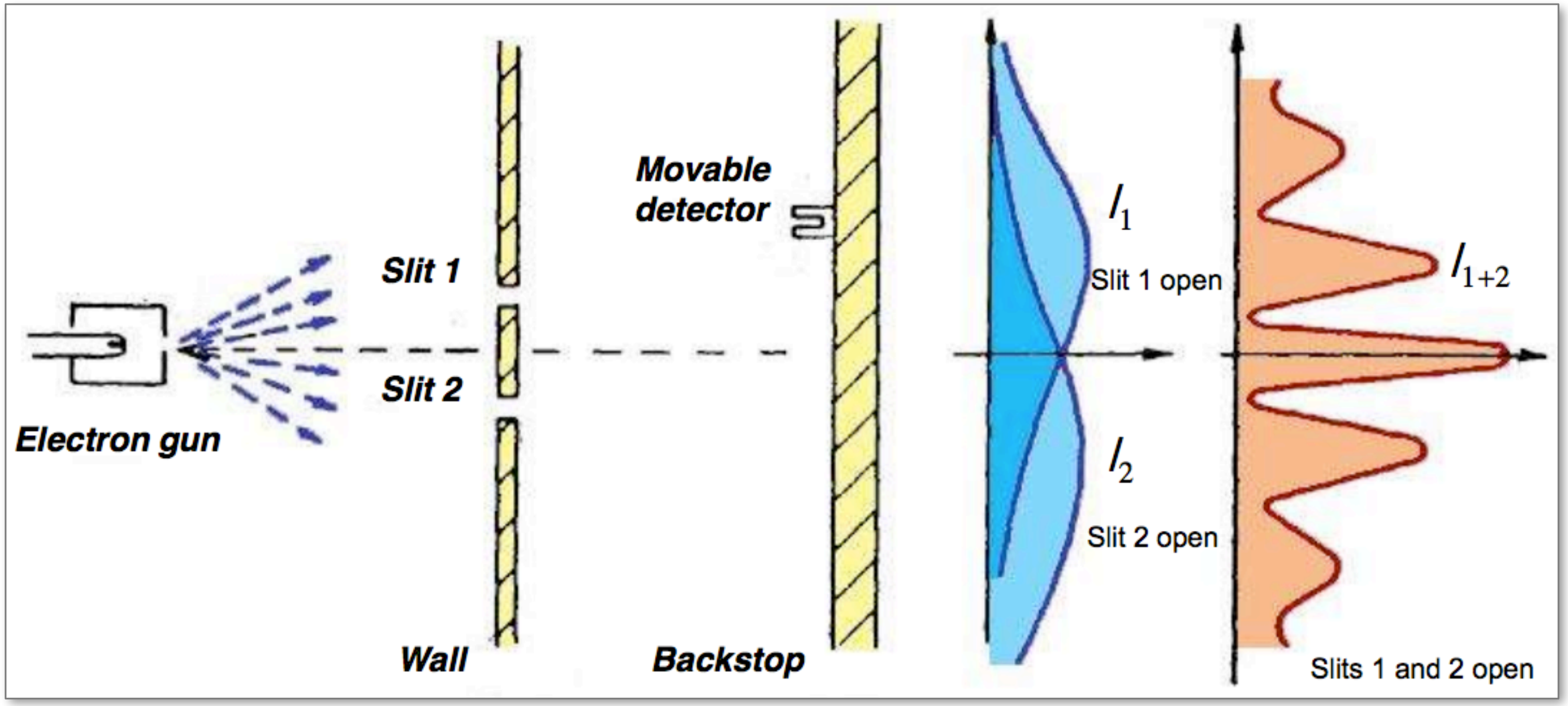
The double slit experiment

- In practice:



- When we perform the real experiment, a strange thing happens.
- When only one slit is open, we get the expected intensity distributions.
- But when both slits are open, a wavelike diffraction pattern appears.
- Apparently, the electrons are acting like waves in this experiment.

Wave function interpretation

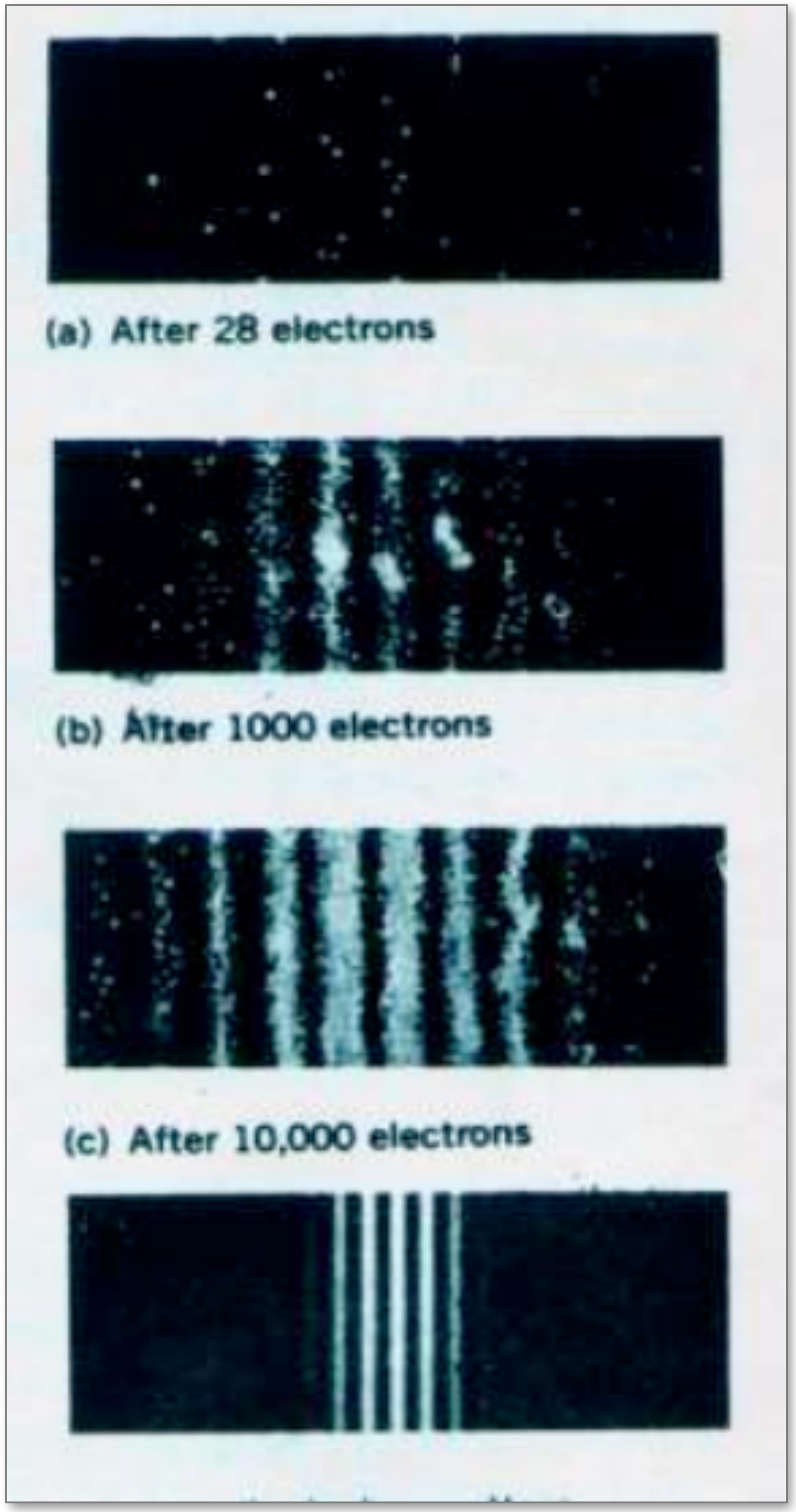


- In terms of waves, the wave function of an electron at the screen when only slit 1 is open is ψ_1, and when only slit 2 is open it's ψ_2.
  - Hence, the intensity when only one slit is open is either $I_1=|\psi_1|^2$ or $I_2=|\psi_2|^2$.
- With both slits open, the intensity at the screen is $I_{1+2}=|\psi_1+\psi_2|^2$, not just I_1+I_2!
- When we add the wave functions, we get constructive and destructive interference; this is what creates the diffraction pattern.

Wave function and probability

- If electrons create a diffraction pattern, they must be waves. But when they hit the screen, they are detected as particles.
  - How do we interpret this?
- M. Born: Intensity $I(x)=|\psi(x,t)|^2$ actually refers to **probability** a given electron hits the screen at position x.
- In QM, we don't specify the exact location of an electron at a given time, but instead state by $I(x)=|\psi(x,t)|^2$ the probability of finding a particle at a certain location at a given time.

The probabilistic interpretation gives physical meaning to $\psi(x,t)$: it is the <u>"probability amplitude" of finding a particle at x at time t</u>.



Diffraction pattern from an actual electron double slit experiment. Notice how the interference pattern builds particle by particle.

Is there a contradiction?

- We now understand the wave function in terms of the probability of a particle being someplace at some time.
  - However, there is another problem to think about.
- Since the electrons diffract, they show wavelike behavior; but when they hit the screen, they interact like particles. And if they are particles, then shouldn't they only go through one slit at a time?
- If this is the case, how can an electron's wave undergo double slit interference when the electron only goes through one slit?
  - Seems impossible…

Is there a contradiction?

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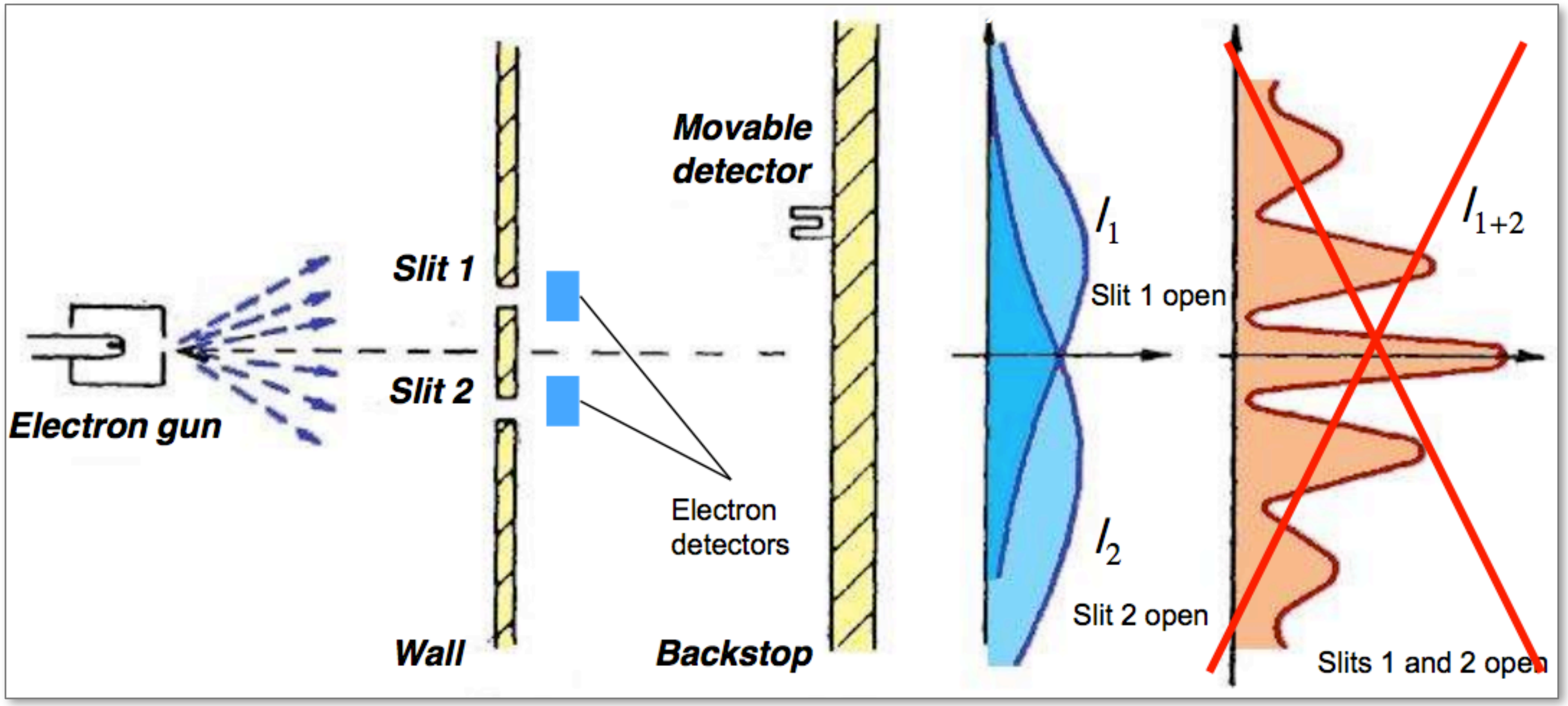
- If this is the case, how can an electron's wave undergo double slit interference when the electron only goes through one slit?
  - Seems impossible…
- To test what's going on, suppose we slow down the electron gun so that only one electron at a time hits the wall.
- We then insert a device over each slit that tells us if the electron definitely went through one slit or the other.

Destroying the interference pattern



- We add electron detectors that shine light across each slit.
- When an electron passes through one of the slits, it breaks the beam, allowing us to see whether it traveled through slit 1 or slit 2 on its way to the backstop.
- Result: if we try to detect the electrons at one of the two slits in this way, the interference pattern is *destroyed*! In fact, the pattern now looks like the one expected for classical particles.

!

Complementarity

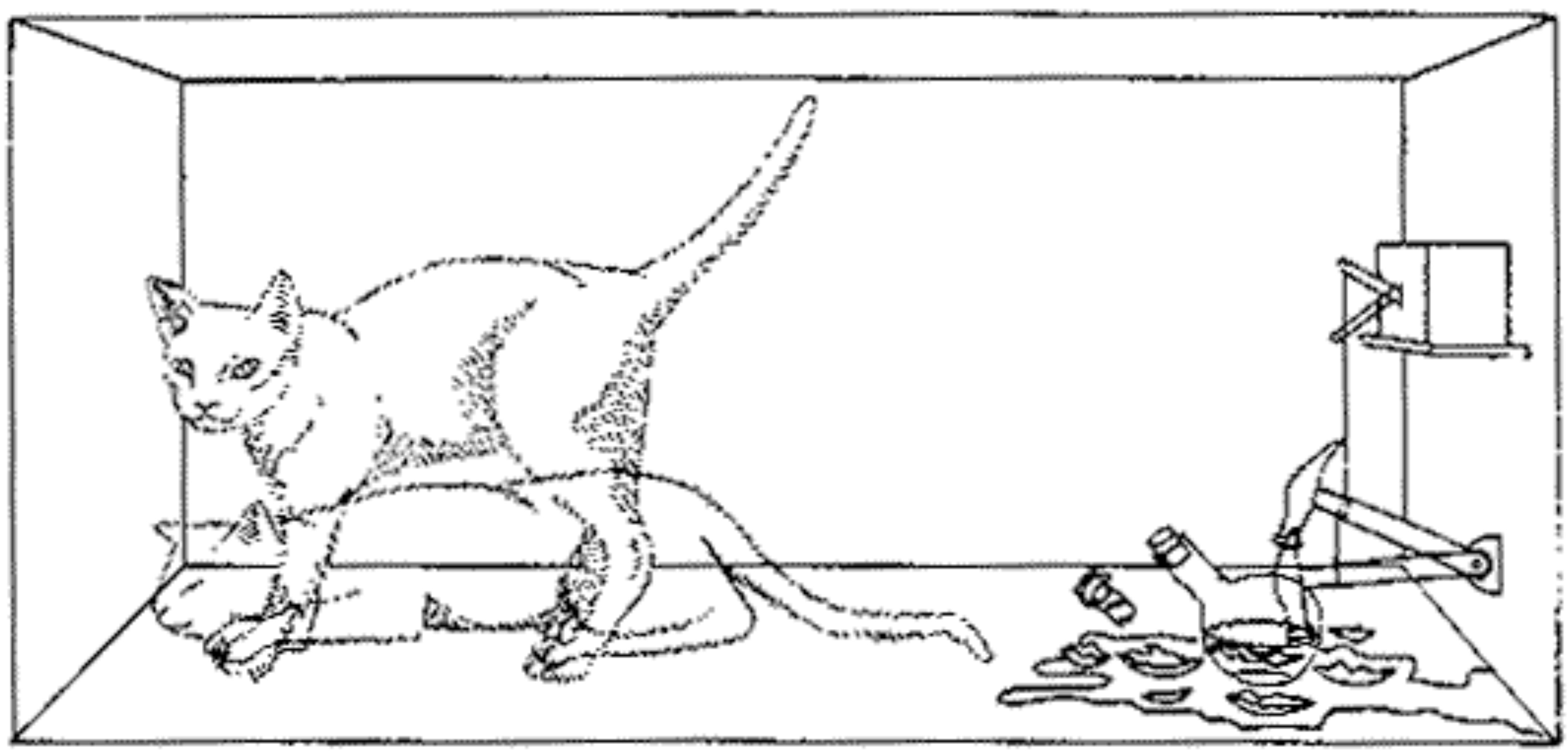
- Why did the interference pattern disappear?
- Apparently, when we used the light beam to localize the electron at one slit, we destroyed something in the wave function $\psi(x,t)$, that contributes to interference.
- Principle of Complementarity (N. Bohr): if a measurement proves the wave character of radiation or matter, it is impossible to prove the particle character in the same measurement, and vice-versa.
- The link between the wave and particle models is provided by the probability interpretation of $\psi(x,t)$, in which an entity's wave gives the probability of finding its particle at some position.

Complementarity

- Why did the
- Apparently, w … ne electron at one slit, we d … $\psi(x,t)$, that contributes t
- Principle of C … nent proves the wave cha … ible to prove the particle c … vice-versa.
- The link betw … vided by the probability in … s wave gives the probabilit

"You changed the outcome by measuring it!"

Schrodinger's cat



Werner Heisenberg, Nobelprize.org

The Uncertainty Principle

- There seems to be some fundamental constraint on QM that prevents matter from acting wave-like and particle-like simultaneously.
- Moreover, it appears that our measurements can directly affect whether we observe particle or wavelike behavior.
- These effects are encapsulated in the Uncertainty Principle.
- Heisenberg: quantum observations are fundamentally limited in accuracy.

The Uncertainty Principle (1)

- According to classical physics, we can (at the same instant) measure the position x and momentum p_x of a particle to infinite accuracy if we like. We're only limited by our equipment.
- However, Heisenberg's uncertainty principle states that experiment cannot simultaneously determine the exact values of x and p_x.
- Quantitatively, the principle states that if we know a particle's momentum p_x to an accuracy Δp_x, and its position x to within some Δx, the precision of our measurement is **inherently** limited such that:

$$\Delta p_x \Delta x \geq \hbar / 2$$

Using the Uncertainty Principle

- Does the Uncertainty Principle (UP) mean that we can't measure position or momentum to arbitrary accuracy?
- No. The restriction is not on the accuracy to which x and p_x can be measured, but rather on the product $\Delta p_x \Delta x$ in a *simultaneous* measurement of both.
- The UP implies that the more accurately we know one variable, the less we know the other. If we could measure a particle's p_x to infinite precision, so that $\Delta p_x=0$, then the uncertainty principle states:

$$\Delta x \geq \frac{\hbar/2}{\Delta p_x} = \frac{\hbar/2}{0} = \infty$$

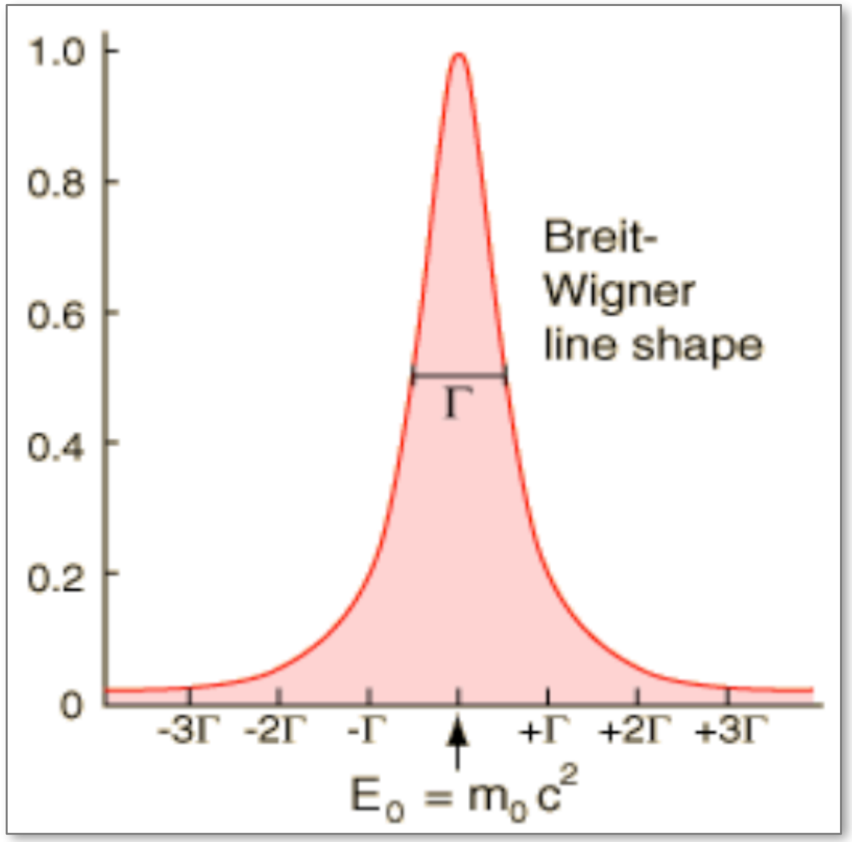
- In other words, after our measurement of the particle's direction (momentum), we lose all information about its position!

The Uncertainty Principle (2)

- The Uncertainty Principle has a second part related to measurements of energy E and the time t needed for the measurements. It states that:

$$\Delta E \Delta t \geq \hbar / 2$$

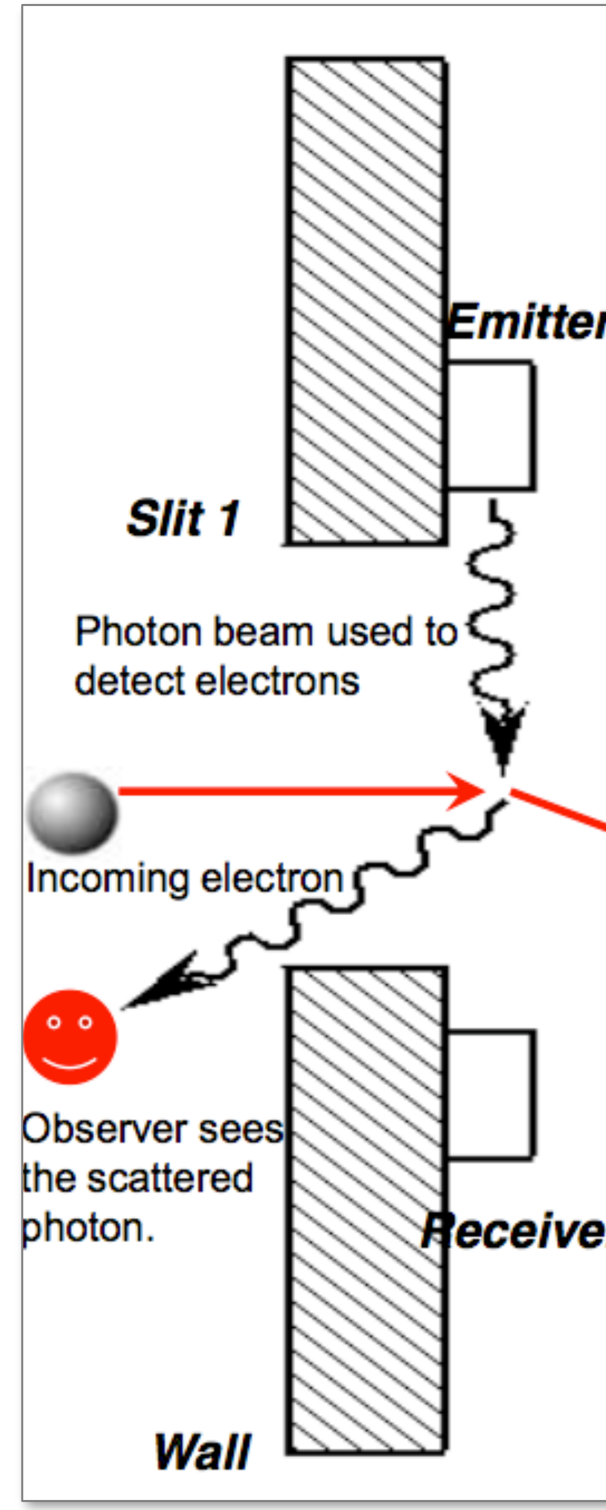
- Example: estimate the mass of virtual particles confined to the nucleus (see Lecture 2 on estimating Yukawa's meson mass).
- Example: Δt could be the time during which a photon of energy spread ΔE is emitted from an atom.
- This effect causes spectral lines in excited atoms to have a finite uncertainty Δλ ("natural width") in their wavelengths.



Atomic spectral lines, the result of transitions that take a finite time, are not thin "delta function" spikes, but actually have a natural width due to the Uncertainty Principle.

Uncertainty and the double slit

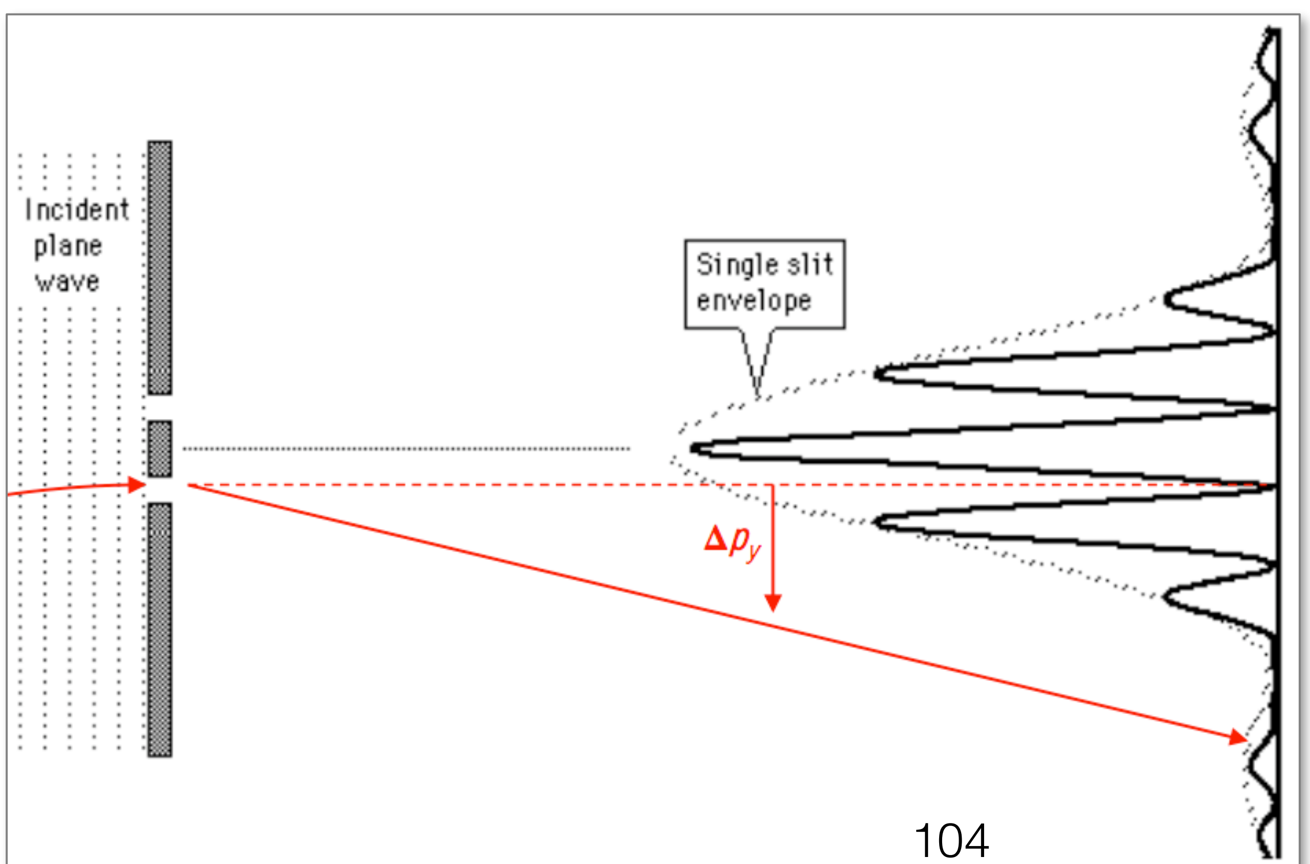
- Now that we know the UP, we can understand why we can't "beat" the double slit experiment and simultaneously observe wave and particle behavior.
- Basically, our electron detector Compton scatters light off of incoming electrons.
- When an electron uses one of the slits, we observe the scattered photon and know that an electron went through that slit.
- Unfortunately, when this happens, the photon transfers some of its momentum to the electron, changing its momentum.



Detection photon Compton scatters off an incoming electron, transferring its momentum.

Uncertainty and the double slit

- If we want to know the slit used, the photon's wavelength must be smaller than the spacing between the two slits.
- Hence, the photon has to have a large momentum (remember, $\lambda=h/p$).
- As a result, a lot of momentum gets transferred to the electron - enough to effectively destroy the diffraction pattern.



If the wavelength of the scattering photon is small enough to pinpoint the electron at one of the slits, the resulting momentum transfer is large enough to "push" the electron out of the interference minima and maxima. The diffraction pattern is destroyed.

Uncertainty and the double slit

- If we try to pinpoint the electron at one of the slits, we change its motion, and the interference pattern vanishes.
- Can you think of a way to get around this problem?

Uncertainty and the double slit

- If we try to pinpoint the electron at one of the slits, we change its motion, and the interference pattern vanishes.
- Can you think of a way to get around this problem?
- We could use lower energy photons…
- It turns out, that just when the photon momentum gets low enough that electron diffraction reappears, the photon wavelength becomes larger than the separation between the two slits (see *Feynman Lectures on Physics*, Vol. 3).
- This means we can no longer tell which slit the electron went through!

There is no way to beat the Uncertainty Principle…

Why the Uncertainty Principle

- Let's review what we have said so far about matter waves and quantum mechanics.
- Elementary particles have associated matter waves $\psi(x,t)$. The intensity of the wave, $I(x)=|\psi(x,t)|^2$, gives the probability of finding the particle at position x at time t.
- However, QM places a firm constraint on the simultaneous measurements we can make of a particle's position and momentum: $\Delta p \Delta x \geq \hbar/2$.
- This last concept, the Uncertainty Principle, probably seems very mysterious. However, it turns out that it is just a natural consequence of the wave nature of matter, for all waves obey an Uncertainty Principle!
- Let's take a look…

Probabilities and waves, again

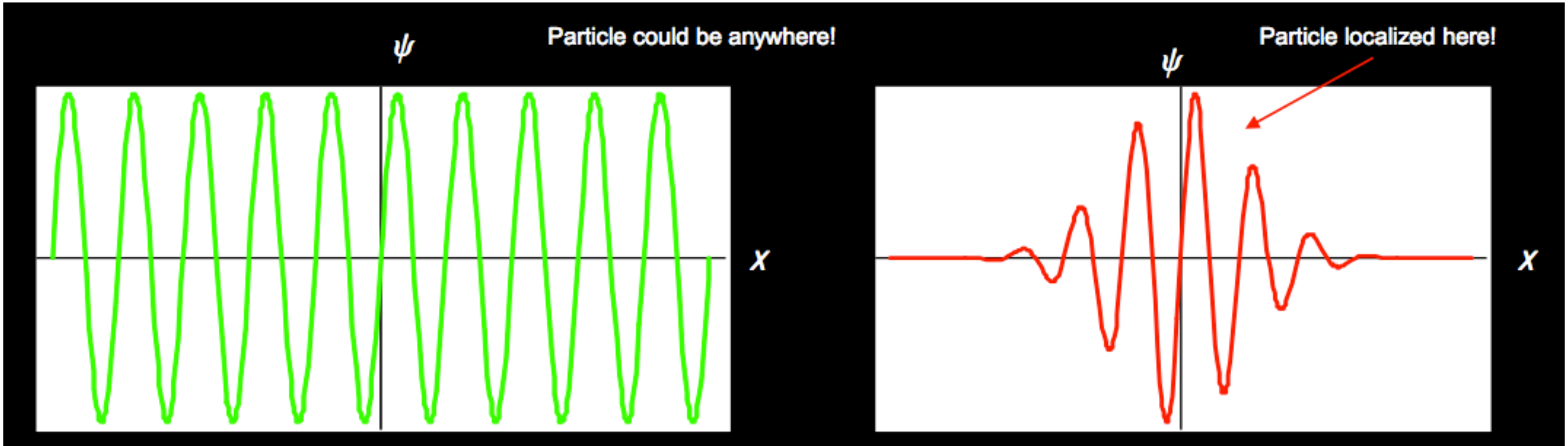
- Consider a particle with de Broglie wavelength λ.
- Defining the wave number $k = 2\pi/\lambda$ and the angular frequency $\omega = 2\pi\nu$, the wave function for such a particle could be (conveniently) written:

$$\psi(x,t) = A\sin(kx - \omega t)$$

- If the wavelength has a definite value λ there is no uncertainty Δλ, and hence $p = h/\lambda$ is also definite.
  - Such a wave is a sinusoid that extends over all values of x with constant amplitude.
- If this is the case, then the probability of finding the particle should be equally likely for any x; in other words, the location of the particle is totally unknown. Of course, this should happen, according to the Uncertainty Principle.

Wave functions should be finite

- If a wavefunction is infinite in extent, like $\psi = A\sin 2\pi(x/\lambda - \nu t)$, the probability interpretation suggests that the particle could be anywhere: $\Delta x = \infty$.
- If we want particles to be localized to some smaller Δx, we need one whose amplitude varies with x and t, so that it vanishes for most values of x. But how do we create a function like this?



Left: a wave function with constant amplitude means that its corresponding particle is equally likely to be at any value of *X*.
Right: a finite wave function, or "wave packet," corresponds to a localized particle with $\Delta X \neq \infty$.

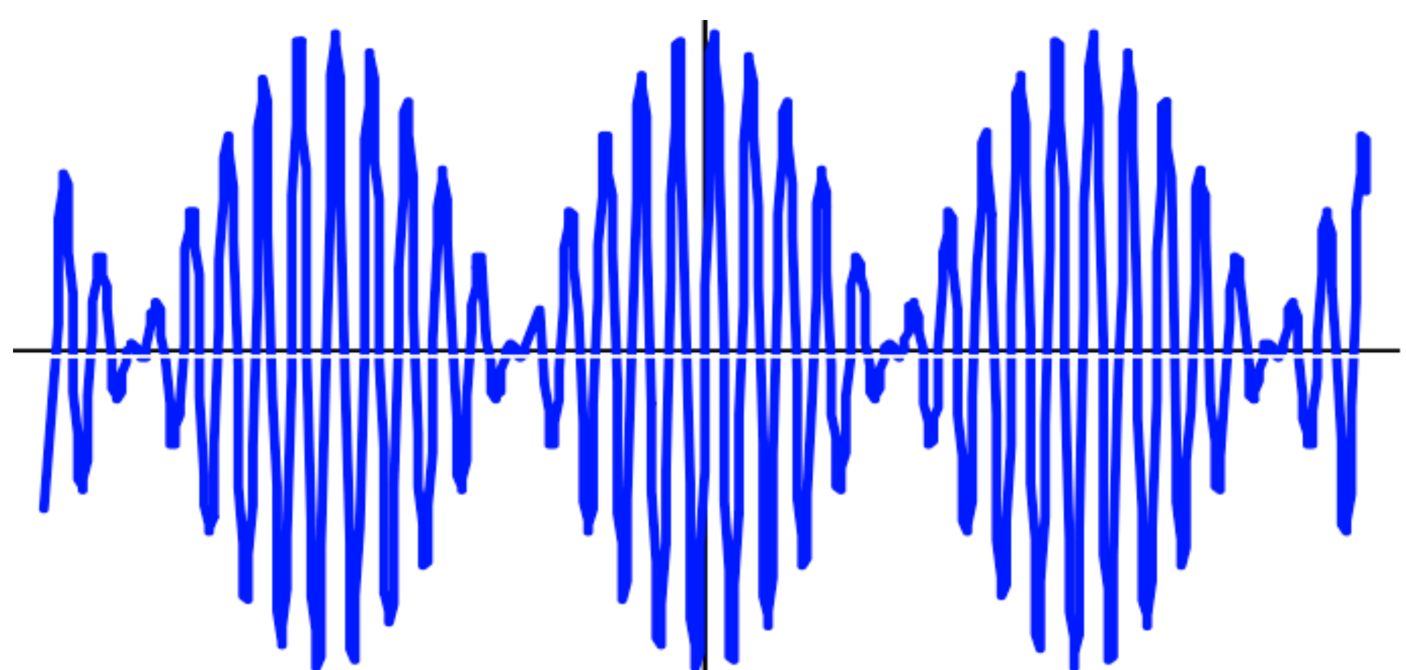
Building wave packets

- In order to get localized particles, their corresponding wave functions need to go to zero as $x \to \pm\infty$. Such a wave function is called a wave packet.
- It turns out to be rather easy to generate wave packets: all we have to do is superimpose, or add up, several sinusoids of different wavelengths or frequencies.
- Recall the Principle of Superposition: any wave ψ can be built up by adding two or more other waves.
- If we pick the right combination of sinusoids, they will cancel at every x other than some finite interval (Fourier Theorem).

Superposition example: beats

- You may already know that adding two sine waves of slightly different wavelengths causes the phenomenon of beats (see below).
- A typical example: using a tuning fork to tune a piano string. If you hear beats when you strike the string and the tuning fork simultaneously, you know that the string is slightly out of tune.

The sum of two sine waves of slightly different wavelengths results in beats. The beat frequency is related to the difference between the wavelengths. This is a demonstration of how adding two sinusoids creates a wave of varying amplitude.



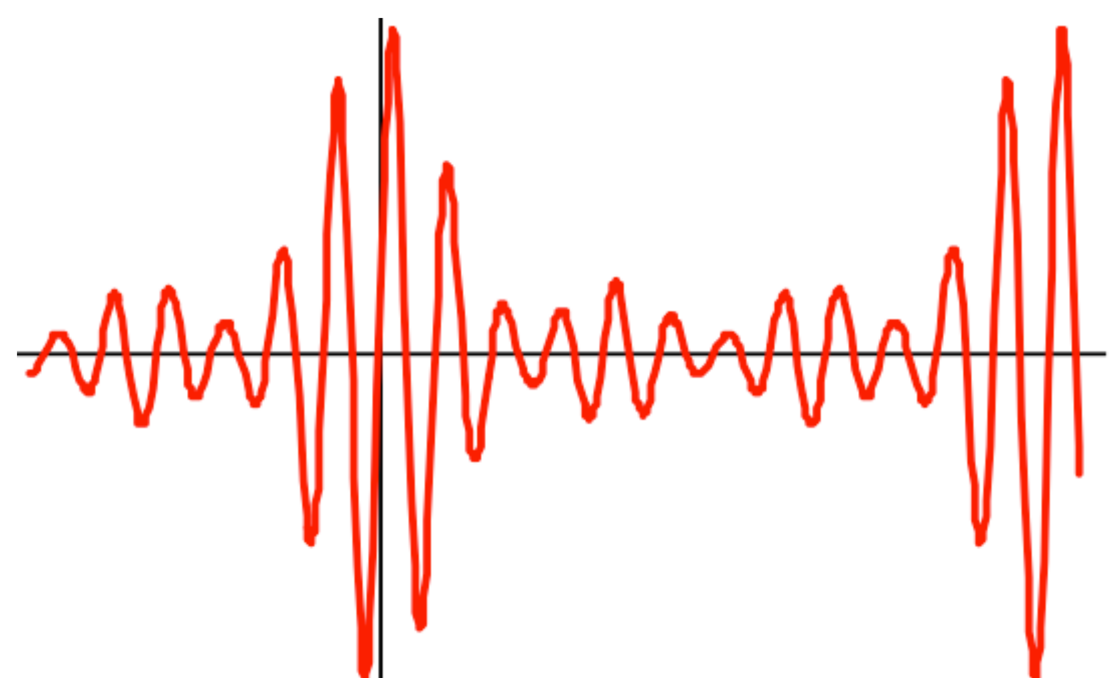
Superimposing more sinusoids

- Now, let's sum seven sinusoidal waves of the form:

$$\psi = \sum_k \psi_k = \sum_{k=9*2\pi}^{15*2\pi} A_k \cos(kx)$$

where $k=2\pi/\lambda$ and $A_9 = A_{15} = 1/4$, $A_{10} = A_{14} = 1/3$, $A_{11} = A_{13} = 1/2$, and $A_{12} = 1$.

- We get a function that is starting to look like a wave packet; however, as you can see, it's still periodic, although in a more complicated way than usual:



Summing up sinusoids to get another periodic function. This is an example of Fourier's Theorem: any periodic function may be generated by summing up sines and cosines.

The continuum limit

- So, evaluating a sum of sinusoids like:

$$\psi = \sum_k A_k \cos(kx)$$

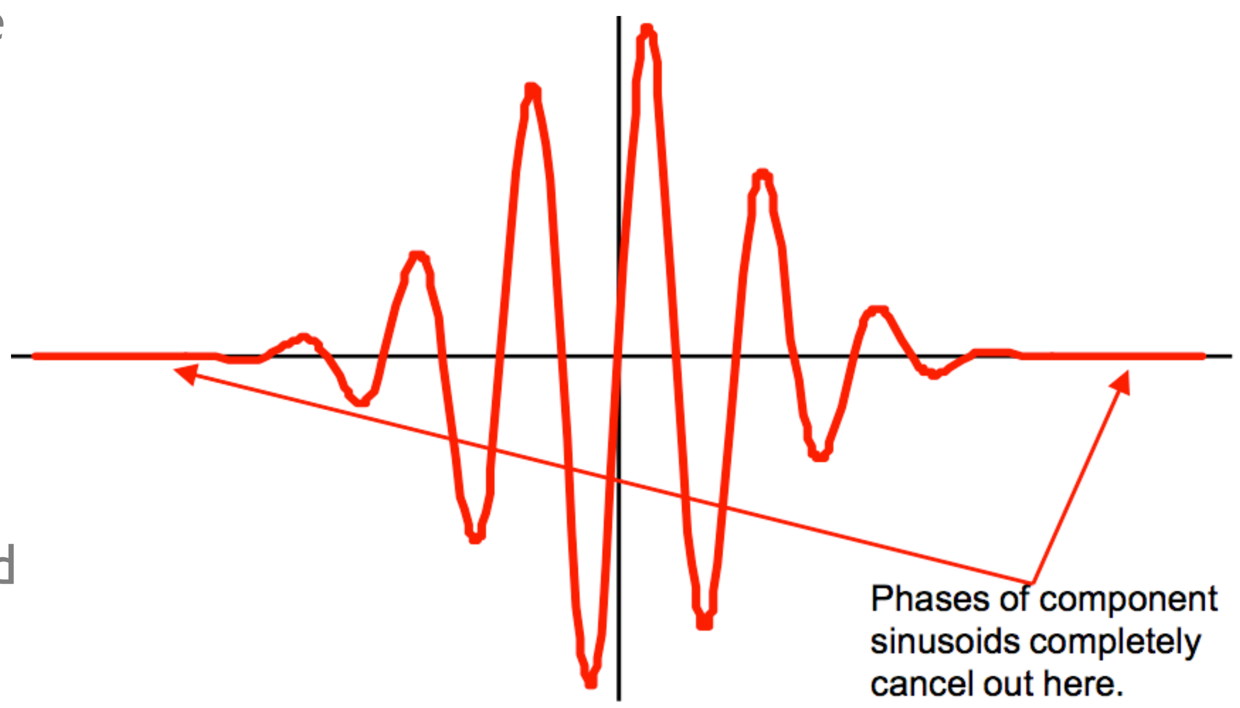
where k is an integer that runs between k_1 and k_2, we can reproduce any periodic function.

- If we let k run over all values between k_1 and k_2 – that is, we make it a continuous variable –then we can finally reproduce a finite wave packet.

By summing over all values of k in an interval Dk= k_1-k_2, *we are essentially evaluating the integral*

$$\psi = \int_{k_1}^{k_2} dk\, A(k)\cos(kx)$$

In such a "continuous sum", the component sinusoids are all in phase near x=0. Away from this point, in either direction, the components begin to get out of phase with each other. If we go far enough out, the phases of the infinite number of components become totally random, and cancel out phases of component sinusoids completely.



Connection to uncertainty

- So, we can sum up sinusoids to get wave packets.
  - What does this actually mean?
- Intuition: we want a wave function that is non-zero only over some finite interval Δx.
- To build such a function, we start adding sinusoids whose inverse wavelengths, $k=2\pi/\lambda$, take on values in some finite interval Δk.
- Here's the point: as we make the interval Δk bigger, the width of the wave packet Δx gets smaller. This sounds like the Uncertainty Principle!

Wave-related uncertainties

- As we sum over sinusoids, making the range of k values Δk larger, we decrease the width of the resulting function.
- In fact, there is a fundamental limit here that looks just like the position-momentum uncertainty relation.
- For any wave, the minimum width Δx of a wave packet composed from sinusoids with range Δk is $\Delta x = 1/(2\Delta k)$, or:

$$\Delta x \Delta k \geq 1/2$$

- There is a similar relation between time and frequency:

$$\Delta t \Delta \omega \geq 1/2$$

Connection to QM

- If $k=2\pi/\lambda$, and $\lambda=h/p$, then the uncertainty relation for waves in general tells us:

$$\Delta x \Delta k \geq 1/2$$

$$= \Delta x \frac{2\pi}{\Delta\lambda} = 2\pi \Delta x \frac{\Delta p}{h}$$

$$\Delta x \Delta p \geq h/4\pi = \hbar/2$$

- We recover the Heisenberg uncertainty relation!
- By a similar argument, we can show that the frequency-time uncertainty relation for waves implies the energy-time uncertainty of quantum mechanics.
- Hence, there is nothing really mysterious about the Uncertainty Principle; if you know anything about waves, you see that it arises rather naturally.

Summary

- Quantum mechanics is the physics of small objects.
  - Its typical energy scale is given by Planck's constant.
- In QM, variables like position, momentum, energy, etc. tend to take on discrete values (often proportional to h).
- Matter and radiation can have both particle and wavelike properties, depending on the type of observation.
- But by the Uncertainty Principle, objects can never be wavelike or particle-like simultaneously.
- Moreover, it is the act of observation that determines whether matter behaves like a wave or a particle.

That's all for this week…

- **Next week:** Experimental methods

Bonus material